AKENTEN APPIAH-MENKA UNIVERSITY OF SKILLS TRAINING AND ENTREPRENEURIAL DEVELOPMENT

IMPROVING STUDENTS' LEVEL THREE OF VAN HIELE GEOMETRIC THINKING ON 3D USING PROBLEM-BASED LEARNING AT ST. JAMES SENIOR HIGH SCHOOL

TECHIE AGYEMANG

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TECHIE AGYEMANG

A thesis in the Department of Mathematics Education, Faculty of Applied Sciences and Mathematics Education, submitted to the School of Graduate Studies in partial fulfilment

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DECLARATION

CANDIDATE'S DECLARATION

I, **Techie Agyemang**, hereby declare that except for reference to other people's work, which has been duly acknowledged, this Thesis consists of my own work produced from research undertaken under supervision and that no part of it has been presented for another degree in this university or elsewhere.

Signature:……………………………….. **Date**: ………………………………

SUPERVISORS' DECLARATION

We hereby declare that the preparation and presentation of this work was supervised in accordance with the guidelines on supervision of Thesis laid down by the Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development.

Professor Ebenezer Bonyah (Principal Supervisor)

Signature:……………………………….. **Date**: ……………………………

Rev. Dr. Benjamin Adu Obeng (Co-Supervisor)

Signature:……………………………….. **Date**: ……………………………

DEDICATION

To my family

ACKNOWLEDGEMENTS

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ABSTRACT

This study aimed at improving students' level three of van Hiele geometric thinking on 3D using problem-based learning at St. James Senior High School in the Sunyani Municipality of the Bono Region in Ghana. The study which was grounded in the positivist paradigm employed a quantitative research approach and a pre-test-post-test quasi-experimental design in analysing the data collected. The intact class sampling was used in selecting two classes of 50 students each for the study. The data collection instrument comprised of pre-test and post-test that was conducted before and after the intervention respectively. Findings from the study revealed a statistically significant difference in the achievement scores of students in the control and experimental groups. Specifically, the study found that students who received instruction based on the PBL and VHPBI recorded high mean sore than those in the control group who received instruction based on the conventional method of teaching geometry. The study therefore recommended that mathematics teachers should be introduced and trained to use the PBL and VHPBI for the effective teaching of geometry.

CHAPTER ONE

INTRODUCTION

1.0 Background to the Study

Mathematics is generally considered the queen of science (Yadav, 2017). While the subject plays a crucial role in advancing various other fields, most students feel apprehensive about it. Mastery of mathematics relies on individual mental capacity. Mathematics offers a way to enhance cognitive abilities and logical thinking, fostering mental agility and creativity (Yadav, 2017). It is an accepted idea in mathematics education that students should understand mathematics (Atepor, 2020). Understanding mathematical concepts usually poses problems to students. According to Atepor (2020), the difficulty of comprehending is inextricably tied to how mathematical knowledge is conceptualized. When students are unable to understand mathematical concepts, their level of success in mathematics falls. According to Atepor (2020), excelling in mathematics holds immense importance for the population of any country. This significance arises from the fact that proficiency in mathematics opens up various educational and professional avenues for students, subsequently enhancing their potential for higher future earnings.

Mathematics has always been associated with geometry (Yadav, 2017). Geometry is a core topic and an important component of the mathematics curriculum, particularly at the Senior High School level (Hassan, Abdullah & Ismail, 2020; Meng & Idris, 2012). Due to the importance of the topic (Geometry), Amidu and Nyarko (2019) suggest that there has been an increasing worldwide focus on the instruction and acquisition of geometry since the mid-1980s. Geometry encompasses the study of shapes and spatial relationships. It unites principles that lay the foundation for algebra, visualizing arithmetic, and statistical concepts (Adelabu, Makgato & Ramaligela, 2019). Exploring geometry imparts fundamental abilities and enhances students' cognitive faculties, encompassing logic, deductive reasoning, analytical thinking, and the aptitude for addressing challenges. Moreover, geometry establishes connections with numerous other mathematical domains like measurement, algebra, calculus, and trigonometry. Geometric principles find practical applications in the routines of architects, engineers, physicists, land surveyors, and various other skilled practitioners (Armah, Cofie & Okpoti, 2018). Geometry knowledge and spatial cognition are required for understanding and enjoying our geometric environment (Alex & Mammen, 2016).

Geometry is crucial to the development of the world. Alex and Mammen (2016) posit that geometry is the mathematics of space and that the aforementioned field of study provides mathematical concepts that enable mathematicians to formulate theories and ideas for the interpretation of space. Through geometry, students study spatial objects, the relationships that exist between these objects and transformations that have been constructed to represent them. Geometric thinking skills are an essential area in mathematics that develops students' critical and analytical thinking in order to gain the necessary skills for 21st-century innovation (Hassan, Abdullah & Ismail, 2020).

Globally, mathematics is recognized as one of the most crucial subjects within the educational curriculum. Despite the potential numerous benefits of geometry to the development of a country. Numerous research findings indicate that geometry poses challenges for many students. (Armah, Cofie & Okpoti, 2018; Alex & Mammen, 2016). Alex and Mammen (2016) posit that students' poor achievement in geometry and mathematics in general is an issue of great concern for many countries across the globe.

The study conducted by Juman, et al. (2022) demonstrated that students in Sri Lanka exhibit limited achievement in the Geometry section of the Mathematics paper in the G.C.E examination. A similar situation was reported by Sulistiowati et al. (2019) who revealed that students in Indonesia perform abysmally in geometry. Sulistiowati, Herman and Jupri (2019) suggest that a common challenge faced by students in geometry lies in solving geometry problems. Students encounter difficulties such as comprehending the provided problem, identifying suitable problem-solving approaches, constructing mathematical models, and executing accurate mathematical procedures (Sulistiowati et al., 2019).

Studies show that to a greater extent, students' poor performance in geometry can be associated with the teaching methods of teachers (Hassan, Abdullah & Ismail, 2020: Meng & Idris, 2012). Research conducted in Ghana shows that mathematics teachers rely on teacher-centred teaching methods such as the lecture method when teaching geometry and other mathematics topics (Armah et al., 2018; Bashiru & Nyarko, 2019). This method of teaching geometry does not afford students the opportunity to have a full grasp of the concepts they are introduced to (Meng & Idris, 2012).

An alternate means of teaching geometry through the van Hiele model has been found to be helpful. In an attempt to find means of improving students' geometric thinking, the van Hiele theory has been used by many researchers. The model identifies many loopholes in the conventional method of teaching geometry in which teachers are seen as repositories of knowledge and students act as mere receptors of concept (Alex & Mammen, 2016). The van Hiele theory opposes the use of teacher-centred instruction. According to the theory, a student's progress in geometry only happens when they have received instruction and this should be student-centred instruction (Todd, 2022). The theory outlines a sequence of five distinct thinking levels that students must advance through step by step, without skipping any level (Hock, Tarmizi, Yunus & Ayub, 2015). The first level is known as level zero. At this stage, students begin to think about shapes and solids based on the way they look. At level one, the first level, students' decision-making relies on visual information. At this stage, their thinking centres on examining a characteristic of a shape. At level two, a student begins to perform a mathematical analysis of solids and give informal justifications based on their properties. At level three, a student is able to write proofs and understand the roles and relationships between axioms, theorems, definitions and postulates. At level four a student is able to make a rigorous analysis of shapes and objects (Todd, 2022).

The van Hiele model offers an understanding of the causes behind the challenges numerous students face when learning geometry. Regarding the learning of shapes and spatial concepts, studies have indicated that students commonly struggle to reach the more advanced stages outlined by the van Hiele model of geometric conceptual development

1.1 Statement of the Problem

Geometry forms an important component of senior high school mathematics curriculum the world over. Despite the importance of the field of study, research has shown that students' performance in the topic has declined over the decade. The attribution of students' poor performance in geometry has been something that is still hanging in serious contention. While some scholars argue that students' poor performance in geometry is a result of their low level of geometric understanding (Amidu & Nyarko, 2019), others also think it is teachers' methods of teaching geometry (Abd Wahab, Abdullah, Abu, Mokhtar & Atan, 2016; Armah, Cofie & Okpoti, 2018)). Abd et al (2016) opine that teachers' methods of teaching geometry are a major factor that impedes students' level of understanding of the concept. They further disclosed that teachers relied solely on the contents of textbooks and taught geometry in an abstract manner. Teachers dissociate geometric concepts from the real-life experiences of students. In addition, Armah et al. (2018) suggest that within the Ghanaian education system, mathematics teachers are regarded as holders of mathematical knowledge, while students are perceived as recipients of mathematical facts, theorems, formulas, and principles. This does not allow students to apply varying levels of thinking in the classroom. This calls for a more student-centred approach to the teaching and learning of geometry.

On the national level, it is evident that geometry is among the topics in mathematics that students face with challenges. For example, the chief examiner's report on mathematics from 2019 to 2021 shows that geometry forms a major component of WASSCE candidates' weaknesses (Chief Examiner's Report, 2019, 2020, 2021). Students' poor performance in geometry is not only revealed in the national average performance during WASSCE as the problem is also evident at St James Seminary Senior High School. Reports from the mathematics department show that geometry is among the concepts that pose challenges to most students. This implies that students perform abysmally in geometry. Personal observations made by the researcher revealed that teachers rely on conventional approaches to teaching geometry and this can be a major factor hindering students' understanding of geometric concepts. Evidence from the literature reveals that teachers' methods of teaching geometry are a major obstacle to students' understanding of geometric concepts (Abd Wahab, Abdullah, Abu, Mokhtar & Atan, 2016; Amidu & Nyarko, 2019; Armah, Cofie & Okpoti, 2018). There is therefore the need for a more student-centred approach to the teaching and learning of geometry.

Globally, the van Hiele levels of geometric thinking are considered a reliable and effective approach for addressing the problems of teaching and learning of geometry. However, prior studies on geometry using the van Hiele levels of geometric thinking have focused on its application in solving two-dimensional geometric problems. To the knowledge of the researcher, it appears that little is known about the application of the van Hiele theory in solving three-dimensional geometric problems. In addition to this little to no research has applied the problem-based learning to the van Hiele theoretical framework to solve geometric problems and improve students' geometric thinking in Ghana. To close this gap; this research work infused the problem-based learning approach in level three of van Hiele's theory to improve St James Seminary Senior High School students' geometric thinking and understanding of three-dimensional (3D) figures to students.

1.2 Purpose of the Study

The aim of this research was to assess how the implementation of problem-based learning, guided by van Hiele Phase-based Instruction, affects students' development of geometric thinking, specifically targeting van Hiele Level 3, with a focus on three-dimensional objects.

1.3 Objectives of the Study

i. To find out the effect of problem-based learning on students' ability to perform geometric proofs on cones and cylinders

ii. To find out the effect of problem-based learning on students' ability to determine the relationships between cones and cylinders.

iii. To determine the effect of Van Hiele theory-based instruction on SHS 3 students' geometric thinking on three-dimensional objects through problem-based learning

1.4 Research Questions

i. What is the effect of problem-based learning on students' ability to perform geometric proofs on cones and cylinders?

ii. What is the effect of problem-based learning on students' ability to determine the relationships between cones and cylinders?

1.5 Research Hypothesis

H0: There is no statistically significant difference among SHS 3 students' performance in level three of van Hiele's levels of geometric thinking, regarding the methods by which they are taught (conventional instruction versus VHPI through problem-based learning).

H1: There is statistically significant difference among SHS 3 students' performance in level three of van Hiele's levels of geometric thinking, regarding the methods by which they are taught (conventional instruction versus VHPI)

1.6 Significance of the Study

This research holds distinct importance as, to the best of the researcher's knowledge, it marks the initial endeavour to assess the efficacy of the VHPI through the utilization of three-dimensional objects within Ghana's Senior High Schools. The individuals who would directly gain from this study include students, math educators, educational institutions, parents, and curriculum designers, among others. Students grappling with challenges in comprehending geometric principles at the Senior High School level in Ghana stand to benefit. Enhancements in geometry teaching strategies would lead to improved academic performance, enhancing students' entry into crucial fields like mathematics, science, and technology. The outcomes of this research would hold considerable significance for educators in mathematics, those involved in creating assessments, and developers of educational curricula. Since the school curriculum plays a crucial role in influencing the quality of education, the insights derived from this study could aid curriculum designers and Senior High School mathematics teachers in effectively implementing the van Hiele model to enhance students' grasp of geometry. Additionally, this knowledge could enhance mathematics teachers' techniques, particularly in the teaching of geometry. The findings that would come out from this study would help improve schools performance in geometry, since an improvement in strategies for teaching geometry will be spelled out. Which can be kept in schools' libraries to serve as reference material for both teachers and students in the teaching and learning of geometry.

1.7 Delimitations

This study was confined to St James Seminary Senior High School. A total of 100 students participated in the study. The study focused on the teaching of the properties, areas and volumes of cones and cylinders. Therefore, the study did not cover all the learning concepts of geometry. The study was restricted to only form two General Science 4 and 7 classes. The study therefore was not extended to all classes of the school. In addition, the study focused on level three of the van Hiele geometry thinking model.

1.8 Definitions of Terms

Van Hiele theory: The Van Hiele theory, crafted by Pierre van Hiele and Dina van Hiele-Geldof in 1957, presents a framework for teaching and learning geometry. It posits that geometric understanding advances through a sequence of five hierarchical stages. The inception of this theory aimed to improve the teaching and learning experience in geometry

Conventional teaching approaches: refer to teacher-centric approaches where geometric information is conveyed to students through lectures, 'chalk and talk' sessions, or by adhering to the sequence outlined in textbooks.

Problem-based learning refers to a student-oriented method where students collaborate in groups to resolve open-ended problems as a means of understanding a subject.

1.9 Organization of the Study

This study was organized into five chapters. The introductory chapter encompassed elements such as the background of the study, statement of the problem, research objectives, research questions, significance of the study, delimitations, limitations, and key definitions used throughout the study. The second Chapter provided an overview of pertinent and related literature in the realm of geometry. This chapter included discussions of conceptual matters, the theoretical underpinnings guiding the study, and reviews of empirical studies. Chapter Three presented the research methodologies employed in the study's execution. In the fourth Chapter of this study, the process of data collection, data processing, interpretation of findings, and subsequent result discussions were outlined. Lastly, the fifth chapter encompassed a summary, conclusions, recommendations, and suggestions for further research.

CHAPTER TWO

LITERATURE REVIEW

2.0 Overview

The current investigation aimed to improve students' level three of Van Hiele geometric thinking on 3D using problem-based learning at St. James Senior High School. This Chapter concentrated on the review of related literature on the subject at hand. Relevant and related literature regarding the issue under investigation are presented in this chapter. The literature comprised of theoretical review, comprises conceptual issues, empirical review and conceptual framework. The literature was retrieved by utilizing databases including ERIC, Hinari, Sage Journals, Google Scholar, Google, and Psycinfo. The chapter was organized under the following sub-headings

1. Theoretical Review
   a. Van Hiele theory of geometric thinking (Pierre & van Hiele, 1957)
2. Conceptual Review
   a. The Concept of Geometry
   b. Geometric Thinking Skill
   c. The Concept of Teaching Geometry in Senior High schools
   d. Reasons for Teaching Geometry in Senior High schools
   e. Difficulties in Learning Geometry among Senior High School students
   f. Strategies to Improve Students Geometric Thinking
3. Empirical Review
   a. Effects of problem based teaching on students' performance in geometry
   b. Effects of Van Heile phase based teaching on students' performance in geometry

4. Conceptual Framework
5. Summary of Literature Review

2.1 Theoretical Review

In this section of the literature, the theory that directed the study's execution was examined. The theory chosen for the study was the Van Hiele theory, which will be elucidated in the subsequent paragraphs along with its relevance to this investigation.

The Van Hiele theory, devised by Pierre Marie van Hiele and Dina van Hiele-Geldof in the 1950s, posits that distinct stages of learning exist to facilitate students' progression from one level of geometric reasoning to a more advanced stage. This model is attributed to the Van Hiele's (Abdullah & Zakaria, 2013). The theory is made up of five stages of geometric cognition that go from level 0 to level 5.

According to Van Hiele (1999), level 0 (Visualization) focuses on the visual component of geometric objects, with pupils beginning to understand the forms of geometrical objects without yet knowing their characteristics. The theory suggests that at this level, students evaluate objects solely based on their appearance. Van Hiele (1999) also emphasized that at level 0, students categorize objects based on their similarity in shape, leading to the creation of classes or groups of shapes that share visual resemblance. Bashiru and Nyarko (2019) suggest that at level 0, a student may recognize geometric shapes such as triangles, cubes, spheres, squares, and circles, but has not yet comprehended their properties. Even though the Van Hiele model outlines characteristics of level 0, individuals at this level may not be aware of these characteristics. At this stage, a person's thinking is heavily influenced by their perception.

According to Van Hiele (1999), level 1 (Analysis) of the Van Hiele theory involves focusing on classes of shapes rather than individual shapes. At this level, students have started to understand the properties of geometric objects and can identify the regularity present within them. However, students at this stage have not yet developed an understanding of the relationships between these geometrical objects. (Machisi & Feza, 2021). At level 1 (Analysis), an individual has acquired knowledge of the properties of the geometric objects they observe. They are capable of recognizing regularities present in these objects. For instance, when looking at a rectangle, an individual can identify that it has two pairs of opposite sides that are parallel to each other. However, at this stage, they have not yet developed an understanding of how the geometry of one object relates to the geometry of other objects.

According to Van Hiele (1999), level 2 (Informal Deduction) of the Van Hiele theory involves focusing on the properties of shapes. At this level, students start to engage in deductive thinking and draw conclusions based on their knowledge of these properties. They begin to sort and determine relationships between objects, and the products of thought are the relationships among the properties of these geometric objects. However, individuals at this level have not yet fully grasped the relationships between the geometry of one object and that of other objects. At level 2, a person has acquired knowledge of geometric shapes and their properties and is able to sequence them in an interconnected manner. For instance, they understand that a square is also a type of rectangle. However, their deductive thinking abilities are just beginning to develop, and they may not be able to explain why the diagonals of a rectangle have the same length.

According to Van Hiele (1999), level 3 (Deduction) encompasses the examination of connections among characteristics of geometric entities. Students in this stage possess the ability to make logical inferences from general principles to specific instances. They also acknowledge the significance of both defined and undefined elements and commence the utilization of axioms or postulates to substantiate various concepts. However, they may not entirely comprehend the rationale behind certain statements being postulates or theorems. Van Hiele (1999) further elaborates that the outcomes of level 3 thinking are deductive axiomatic systems for geometry. During this phase, individuals grasp the appropriateness of deduction as a means of constructing geometry within axiomatic frameworks. They must assemble the proof rather than merely accepting it. The architecture of a comprehensive axiom system involving axioms, definitions, theorems, consequences, and postulates becomes the explicit focus of their reasoning. A variety of proof methods can be employed, and they can differentiate between assertions and arguments. At the deduction level, it becomes evident that the diagonals of a square possess identical lengths, and the necessity of establishing this assertion through a sequence of deductive justifications is recognized.

According to Van Hiele (1999), level 4 (Rigor) is characterized by regarding deductive axiomatic systems for geometry as subjects of contemplation. In this phase, students recognize the significance of precision in fundamental principles during proofs, such as Euclid's geometry postulates. The phase of rigor entails intricate and demanding cognitive processes, which might explain why high school students might not yet have reached this stage of cognitive development. As described by Van Hiele (1999), the outcomes of thought at level 4 involve comparisons and distinctions among diverse axiomatic systems of geometry. At the rigor stage, an individual is proficient in navigating multiple axiomatic systems, even including non-Euclidean geometry. It can be asserted that an individual at the rigor stage has advanced through levels 0 to 3 and has cultivated the capacity for reversible thinking, rendering them more adept at anticipation and exploration—both of which are conducive to achieving the requisite cognitive depth for problem-solving.

2.1.1 Van Hielle Phase Based Instruction

To ensure successful teaching of geometry, educators must tailor their approach based on their students' current level in the Van Hiele model (Armah et al., 2018). The Van Hiele model outlines five phases of learning that students typically go through in sequence (phase-based instruction), and teachers can use this model to help their students progress from one level to the next.

Phase 1, Information. During this phase, the teacher initiates conversations with the students about the topic to be studied, assesses their responses, and creates an understanding of the purpose of studying the topic to prepare for further learning (Atepor, 2020).

Phase 2, referred to as Guided Orientation, entails students participating in hands-on activities specifically crafted to acquaint them with the tangible objects from which geometric concepts are derived. The teacher guides students as they actively explore the topic of study and perform simple tasks to elicit specific responses. (Battista, 2011).

Phase 3, known as Explicitation, involves helping students articulate their understanding of geometric structures discussed in class. In this phase, the teacher guides class discussions until a consensus is achieved, ensuring that students gain a clear understanding of the subjects under study. Following this, the teacher introduces relevant terminology to facilitate more precise communication of students' ideas (Battista, 2011; van Hiele, 1999).

During the fourth phase, referred to as Free Orientation, the teacher assigns students more intricate and multifaceted tasks that allow for various approaches and solutions (van Hiele, 1999). Students are encouraged to solve these problems and to discuss and elaborate on their solution strategies. The teacher also introduces relevant problem-solving processes as needed (Battista, 2011).

Phase 5, referred to as Integration, involves students consolidating their learning about the topic by synthesizing their understanding and developing a comprehensive overview of the subject matter. The aim is to attain a new level of understanding by the end of this phase, which prepares them for the next level of learning, where they can repeat the five phases of learning (van Hiele, 1999).

In relation to this study, there is the need for teachers to determine the level of theory attained by leaners. Knowing this will serve a long way in determining learners entry behaviour thereby knowing the effective interventions to guide learners in their learning process. In addition, there is the need for teachers to know which level learners have attained because as proposed by the study a learner ought to attain level zero before he/she can progress to level one, two and so on. Moreover, it can be said that students' geometric thinking and performance are likely to improve when they are taken though the various phases of learning proposed by the Van Hiele theory. The Van Hiele Theory has the potential to improve students' geometric thinking when employed in the teaching of geometry.

2.2 Conceptual Review

This section defines some of the terms or variables utilized in the study. The next paragraphs elaborate on the topics employed in the study.

2.2.1 Problem-Based Learning

The Problem-Based Instruction (PBL) model is an educational approach that involves students in solving real-world problems as a means of constructing their own understanding, developing critical thinking skills, and building independence and self-confidence (Reski, Hutapea & Saragih, 2019). This model prioritizes problem-solving as a learning process and involves a series of educational activities (Simamora, Sidabutar & Surya, 2017). According to Sulistyowati, Budiyono, and Slamet (2017), PBL encourages students to do research, bridge the gap between theory and practice, and use their knowledge and abilities to create practical solutions to particular issues.

According to Yoo (2008), Problem-Based Learning focuses on students' learning processes. Unlike traditional teaching methods where teachers are the main source of knowledge, PBL prioritizes students by providing them with opportunities to prove theorems and solve problems. This approach helps students to develop critical and analytical thinking skills and to effectively use educational resources. PBL fosters an environment where students are encouraged to build upon their existing knowledge to construct new understanding. This method fosters skill development, critical thinking, and directs student involvement in the learning process through authentic activities. Problems serve as the starting point for learning in the PBL model, and existing problems form the basis for learning. (Dewi & Wardani, 2018; Rahmawati et al., 2023; Rahmi et al., 2022).

In order to cultivate a learning approach centred around students, the Problem-Based Learning (PBL) model promotes collaborative group work where students collaborate to address real-life challenges, thereby sparking their curiosity and honing their analytical skills. The PBL method fosters critical and analytical thinking in students and teaches them how to effectively locate and use resources to enhance their learning experience. (Yulianti & Gunawan, 2019).

Rahmawati et al (2023) proposed five phases that teachers need to follow for effective implementation of the PBL model. The first phased requires teachers to pose a problem (question). They advocate that such question(s) should be unstructured such that additional information is required to complete them. The second phase involves asking "what is known from the problem?" This phase requires students to approach the problem from their already existing knowledge (previous knowledge on the problem). At this phase members of the group collaborate to explore issues and aspects of the problem that can be in further investigations. This phase serves as the starting point towards finding a solution to the problem. The third phase involves asking what is not known about the problem. At this phase members of the group analyse the problem from various angles by breaking it down into smaller components. Members then discuss the implications of the problem, propose explanations or solutions from varies angles, and formulate working hypotheses. This stage requires learners to employ critical thinking as well as analytical skills in addressing the problem. The fourth stage of the process, called "Alternative Solutions," involves group members discussing, evaluating, and organizing different hypotheses, as well as modifying existing hypotheses. During this stage, the group assigns tasks and develops plans for obtaining necessary information. The fifth stage is called "Report and Presentation of Results," during which each group creates a report about their work. The teacher uses this report to develop further material for in-depth learning based on the concepts proposed by each group. Additionally, during this stage, the group's report undergoes material development (Handayani, 2017).

Chaudhari and Rodrigues (2016) propose that the Problem-Based Learning (PBL) procedure involves the following stages: a) Introducing students to problems: The teacher acquaints students with the problem, outlines the learning goals, clarifies logistical details, and inspires them to partake in problem-solving activities. b) Structuring student learning; The teacher assists students in delineating and structuring pertinent learning activities associated with the problem. c) Guiding individual/group exploration: The teacher guides students in navigating their individual or collaborative experiences, encouraging them to collect appropriate information, conduct experiments, and create solutions. d) Developing and presenting work: The teacher helps students in the planning and preparation of their tasks, such as reports, and supports the sharing of assignments among their peers. e) The teacher aids students in examining and evaluating their problem-solving process by fostering reflection on their investigations and the techniques they have used.

2.2.2 The Concept of Geometry

Geometry is a vital area of mathematics that focuses on the properties of spaces (Uduosoro, 2011). In its simplest form, geometry concerns the measurement of the areas and diameters of two-dimensional shapes, as well as the surface areas and volumes of three-dimensional solids. Descriptive geometry, topology or analysis situs, geometry of spaces and non-Euclidean geometry are other fields of geometry.

According to Schwartz and Heiser (2006), geometry is the study of shapes and sizes of figures, which is known as plane geometry when dealing with two-dimensional shapes and analytical geometry when using algebra and coordinates (numbers) to solve geometric problems. Euclidean geometry, based on Euclid's postulates, especially parallel postulates, is known as Euclidean geometry, while non-Euclidean geometry is based on different sets of postulates to create a consistent system. Pilant (2009) stated that geometry deals with the properties of space and that high school students study plane geometry, which focuses on flat surfaces, before moving on to solid geometry, which focuses on three-dimensional shapes. Pilant also noted that geometry has many other fields, including the study of spaces with four or more dimensions.

Uduosoro (2011) stated that the term geometry is an accurate representation of the work of early geometers, who were primarily concerned with practical problems such as measuring field sizes and ensuring that corners of buildings were accurately right-angled. This empirical geometry was first developed in ancient Egypt and was later refined and systematized by the Greeks. Pythagoras is credited with laying the foundation for scientific geometry by demonstrating that the arbitrary and disconnected laws of empirical geometry could be derived from a limited set of axioms or postulates. These postulates were considered by Pythagoras and his followers to be self-evident truths, but in modern mathematical thinking, they are viewed as a set of convenient but arbitrary assumptions (Battista, 2011).

Amidu and Nyarko (2019) noted that geometry and its concepts play a significant role in the creation of our synthetic world, including fields such as arts, architecture, and automobiles. Studying geometry provides students with valuable skills in deductive reasoning, visualization, problem-solving, proof, critical thinking, logical argument, and other lifelong skills. According to Alex and Mammen (2016), geometry is a crucial contributor to the advancement of the world. They argue that it is the mathematics of space and provides mathematicians with essential concepts to develop theories and ideas for interpreting space.

Hassan, Abdullah, and Ismail (2020) state that the study of geometry involves the examination of spatial objects, their relationships, and the transformations used to represent them. The development of geometric thinking skills is a crucial component of mathematics education that enhances students' critical and analytical thinking, enabling them to acquire the necessary competencies for innovation in the 21st century. Geometric objects are widely employed across diverse professional domains. For instance, architects and civil engineers utilize points to formulate architectural blueprints, while professionals in Computer-Aided Design (CAD) utilize line segments to draft intricate mechanical components (Hollebrands & Stohl Lee, 2011). Furthermore, geometry often involves delving into theoretical notions, like dimensionless points or infinitely extending one-dimensional lines. These entities are exclusively conceivable in the realm of imagination. Geometry fundamentally hinges on visualization, as it's challenging to engage in geometric thinking devoid of sketching or utilizing visual aids to depict abstract geometric concepts. For instance, engineers in technical fields frequently produce visual diagrams of their proposed undertakings as a means to convey their concepts to others (Velichova, 2002).

2.2.3 Geometric Thinking Skill

Geometric Thinking (GT) refers to a specific type of mathematical thinking that is focused on a particular area of study. This kind of thinking is crucial for educators to instil in their students. Visualization is a critical component of GT, allowing learners to mentally manipulate and view objects from multiple perspectives. Developing GT is essential to helping students improve their critical thinking abilities, and it is a valuable branch of mathematics.

Skrbec and Cadez (2015) suggest that there are various factors that affect the ability of students to learn geometry effectively, one of which is their capacity to think in a geometric manner. Furthermore, if students show more interest in studying geometry, it can enhance their geometric thinking ability. According to Van Hiele's theory, the development of geometric thinking in students is influenced more by their experience in learning geometry rather than their biological maturity or grade level. Insufficient exposure to geometry education results in limited learning experiences (Armah et al., 2018). Skrbec and Cadez (2015) state that Van Hiele's study has led to the creation of a model that outlines five levels of geometric thinking, which are arranged in order of significance. The five levels are visualization, abstraction, analysis, deduction, and rigor. Teachers must have a thorough understanding of their students' level of geometric thinking. The geometry instruction provided to students should align with their current level of thinking so that they can progress to the next level (Mammarella et al., 2017).

As stated by Trimurtini et al. (2021), the spatial aptitude and level of geometric comprehension among students serve as essential factors that educators should consider when initiating their instruction. When students exhibit varying degrees of geometric thinking, teaching approaches should be adapted to align with their individual levels. Consequently, while teaching the same geometric concept, the approach to conveying the information can be tailored in diverse manners, in accordance with each student's specific level of geometric understanding. The objective is to assist students in progressively attaining mastery across all stages of geometric thinking.

In earlier studies, it was found that deductive reasoning has not been attained by several teachers or pre-service teachers (Denizli & Erdoğan, 2018). Various studies on geometric thinking levels with mathematics pre-service teachers have indicated that most of them are at level 3 (Fitriyani et al., 2018) while achieving level 4 is infrequent and challenging.

Furthermore, Hohol (2020) conducted a more detailed examination of various phases in the development of geometric thinking. The initial phase revolves around visualization, in which students perceive geometric shapes primarily as visual entities, without delving into their constituent elements, components, or geometric attributes. The subsequent stage is termed descriptive or analytic, where students progress beyond mere visual recognition of shapes and start comprehending their inherent characteristics and the relationships existing among them. The third level, known as abstract or relational, denotes students' ability to understand the hierarchical relationships between geometric characteristics and concepts, as well as their fundamental and necessary requirements. The next level is formal deduction, in which students learn definitions, axioms, theorems, and proofs. The highest level is meta-mathematical, which denotes geometers who have a thorough knowledge of the interactions between Euclidean and non-Euclidean geometries.

2.2.4 The Concept of Teaching Geometry in Senior High schools

As outlined by the National Council of Mathematics, USA (NCTM) (2000), formal geometry is recognized as the most intricate topic within the high school mathematics curriculum. This differentiation primarily stems from its deductive structure, setting it apart from other mathematical domains and posing teaching challenges. The NCTM, an international association of mathematics educators, also recognizes that formal geometry is a complex field, offering difficulties in both teaching and understanding because students often have a lower level of geometric thinking than what is required by this subject.

According to Uduosoro (2011), students often find geometry to be a tedious subject because it is primarily focused on learning proofs, which they may find difficult to comprehend. This may be due to a lack of emphasis on developing fundamental skills necessary for learning geometry. Hoffer lists five crucial skills that should be incorporated into the geometry curriculum, including logical, drawing, visualization, verbal, and applied skills.

During the late 1960s, the primary role of mathematics teachers in American classrooms was to provide examples, state rules, and present problems, as stated by the National Council of Teachers of Mathematics (NCTM, 1970, p. 21). Unfortunately, according to Uduosoro (2011) the teaching of mathematics, including geometry, remained unchanged.

To address this, the National Council of Teachers of Mathematics in the United States of America suggests using diverse teaching methods to encourage students to approach geometry learning in a self-directed and imaginative manner.

Geometry has been a contentious subject in the mathematics curriculum for secondary school students. The National Assessments of Educational Progress indicate that students' performance and attitudes towards geometry are low. Recent studies suggest that teachers' instructional methods can affect students' attitudes towards mathematics, especially geometry (Uduosoro, 2011). According to the literature, secondary school students should be able to demonstrate geometric thinking at Van Hiele Levels 3 and 4, but research shows that they are below the expected level (Fabiyi, 2017; Luneta, 2015).

There are several arguments in favor of including geometric proof into the high school mathematics curriculum. Proofs enable students to improve their problem-solving, visual comprehension, deductive reasoning, critical thinking, and analytical abilities in addition to laying the groundwork for future endeavors in Mathematics, Science, and Technology (Ndlovu & Mji, 2012). These abilities are critical in promoting the general intellectual growth of a nation's people.

Despite the many reasons for incorporating geometric proofs in the high school mathematics curriculum, reports from examiners in various countries indicate that students struggle with Euclidean geometry questions that require them to construct non-routine proofs, and many students do not even attempt such questions (Mwadzaangati, 2015; West African Examination Council, 2015).

According to Battista (2011), students tend to dislike geometry due to their difficulty in comprehending the subject and its proofs. Students typically view geometry as a tedious, stressful, and challenging subject in which they must prove abstract axioms, theories, and postulates. This means that the way in which geometry is presented to students has an impact on their interest and performance in the subject. As Sullivan and Glanz (2004) suggested, instruction is a complex process that involves students' motivation, cognition, and learning processes. Teachers' planning of instruction plays a significant role in shaping students' achievement in a particular subject. Therefore, teachers should employ instructional principles when developing lesson activities.

Jones, Fujita and Ding (2006) carried out research to investigate effective ways of teaching and learning geometry, particularly in relation to developing students' deductive reasoning. They examined teaching methods utilized in China and Japan and proposed that studying geometry should include applying geometric principles in multiple situations through modeling, problem-solving, and deductive reasoning. They found that effective geometry teaching can be improved by implementing good pedagogical models that are supported by well-designed learning tasks and tools.

2.2.5 Reasons for Teaching Geometry in Senior High schools

Geometry is a crucial subject to study as it provides students with a range of academic and career prospects, and is often used as a criterion for admission to higher education programs and professional work opportunities (Anamuah-Mensah, 2007). Mastery of geometry is essential for individuals pursuing fields such as physics, engineering, and technology, and is a crucial stepping stone for educational and career advancement. Within the field of mathematics, Geometry distinguishes itself with intriguing challenges, rich historical roots, and strong connections to the evolution of mathematics, rendering it an essential component of the discipline (Jones, 2014). Geometry is a major topic in mathematics that must be taught in schools, and It's no surprise that learners of all levels confront it (CRDD, 2012). Geometry is a powerful tool that allows students to analyze and interpret the world around them while also providing them with skills that can be applied to other areas of mathematics (Funny, Ghofur, Oktiningrum, & Nuraini, 2019).

The objectives of instructing geometry, as outlined in the mathematics syllabus (CRDD, 2012), encompass: cultivating spatial awareness, refining geometric intuition and visualization abilities, offering diverse geometrical encounters, nurturing comprehension of geometrical properties and theorems, fostering the utilization of conjecture, deductive reasoning, and proofs, amplifying problem-solving proficiencies in real-life scenarios through geometry's application, honing ICT skills pertaining to geometry, cultivating a favourable disposition toward mathematics, raising awareness about geometry's historical and cultural relevance within society, and showcasing the present-day applications of geometry.

As Jones (2014) suggests, the incorporation of Geometry into the educational curriculum is promoted for its ability to nurture skills like critical thinking, problem-solving, perspective, visualization, conjecture, deductive reasoning, proof, logical argument, and intuition, among students. Furthermore, Geometry contributes to improving understanding in diverse mathematical realms. Jones emphasizes the importance of delivering Geometry in a way that sparks curiosity and encourages exploration, ultimately enhancing students' learning journeys and shaping their broader outlook on both Geometry and Mathematics.

Jones (2014) argues that the teaching of geometry at the senior high school level holds importance due to its profound visual presence in our cultural sphere. Numerous cultural elements, ranging from artefacts and arts to architecture and music, incorporate geometric concepts like symmetry, perspective, and scale. Moreover, scientific and technological concepts hinge on geometry and various contemporary applications of mathematics rely on it. Thus, the significance of geometry across diverse fields cannot be emphasized enough.

2.2.6 Difficulties in Learning Geometry among Senior High School Students

There has been a growing concern regarding the geometric thinking abilities of students in Ghanaian schools, particularly at the primary school level, as noted in several studies (Armah et al., 2018; Baffoe & Mereku, 2010; Mullis et al., 2011). Furthermore, the yearly reports of the West African Examination Council's (WAEC) Chief Examiners, spanning from 2010 to 2021, noted that students exhibited inadequacy in comprehending Geometry of circles and 3-dimensional problems. The reports stated that a majority of the students opted not to tackle questions related to 3-dimensional problems, while only a few candidates demonstrated a proper understanding of the issue in their responses, despite attempting Geometry questions (Chief Examiner's Report, 2010, 2019, 2020, 2021). Despite the significance of mathematics, students have not been performing well in this subject. The West African Examination Council's (WAEC) Chief Examiner's report of 2021 noted that numerous students avoid geometry problems throughout their West African Senior High School Certificate Examinations (WASSCE).

Mathematics educators have made considerable efforts to identify the issues related to the teaching and learning of mathematics in classrooms. However, despite these persistent endeavours, poor mathematics performance continues to affect public examinations in the country (Adolphus, 2011). Mullis et al (2011) report that Ghanaian students' performance in mathematics indicated weaknesses in algebra and geometry.

Mathematics, especially geometry, is challenging to learn and teach and is commonly associated with poor performance (National Mathematics Centre, 2009). These difficulties in geometry stem from various factors, including teachers, students, and environmental factors.

A vast majority of students harbor a strong aversion towards geometry due to the ineffective way it was initially taught, which led to their inability to grasp the subject's concepts or its significance. The lack of emphasis on these issues in the curriculum for geometry teacher preparation may explain this failure, as suggested by Helena and Maria (2015). Additionally, this problem may stem from the fact that geometry is not taught at a level that aligns with the learners' cognitive abilities. Miller (2011) proposed that instructors should assess the student's current cognitive level, as well as their strengths and weaknesses, before commencing a lesson with them.

Kivkovich (2015) suggests that students' challenges in geometry stem from their inability to comprehend mathematical language in geometry and relate it to their previous knowledge. Other factors contributing to students' struggles in learning geometry have been identified by Gloria (2015). These factors include a poor grasp of fundamental concepts, inadequate comprehension of mathematical skills, and students' disinterest in the subject, all of which can lead to difficulties and unfavourable reactions towards the subject matter. Veloo & Diah (2014) state inadequate teaching approaches and a lack of instructional materials are some of the factors that contribute to students' struggles in learning geometry.

Gebremichael (2014) believes that students' low motivation to engage in mathematics is another reason for poor performance. Additional factors include the absence of instructional materials, a lack of reasoning skills, insufficient time, gender disparities, an unsuitable curriculum, and students' inability to provide proof (Uduosoro, 2011; Nigerian Educational Research and Development Centre, 2012). Studies show that to a greater extent, students' poor performance in geometry can be associated with the teaching methods of teachers (Hassan, Abdullah & Ismail, 2020: Meng & Idris, 2012). Research conducted in Ghana shows that mathematics teachers rely on teacher-centred teaching methods such as the lecture method when teaching geometry and other mathematics topics (Armah et al., 2018; Bashiru & Nyarko, 2019). This method of teaching geometry does not afford students the opportunity to have a full grasp of the concepts they are introduced to (Meng & Idris, 2012).

According to the Van Hiele theory, students can effectively grasp geometry only by progressing sequentially through all levels, without omitting any. To excel at level (n), students must first attain mastery at level ($n-1$) (Usiskin, 1982). The Van Hieles theory indicates that a significant number of secondary school students struggle to learn and comprehend geometry due to teachers introducing geometry concepts at a level surpassing the students' current understanding (Van Hiele, 1984). This disparity between instruction and comprehension creates a mismatch. The Van Hieles cautioned against compelling students to attain a particular level prematurely, as this could lead to students merely replicating the teacher's actions without true comprehension (Van Hiele, 1999).

2.2.7 Strategies to Improve Students Geometric Thinking

According to the cognitive load theory, effective learning happens when it conforms to the organization and capabilities of human cognitive processes. This involves considering the requirements for working memory during activities like learning, reasoning, addressing problems, or forming mental images (Atepor, 2020). For instance, the way information is presented can hinder learning by causing an external cognitive load, which might be excessive or duplicative. Introducing a three-dimensional geometric scenario using a three-dimensional illustration, for instance, could create an unnecessary external cognitive load for certain students, potentially hindering their thinking and learning processes (Sulistiowati et al., 2019). To attain an optimal level of learning, it is crucial to reduce and control the external cognitive burden stemming from how information is presented. Instead, the focus should be on increasing the cognitive load related to structuring information into mental frameworks (Hassan et al., 2020).

Conversely, Schwartz and Heiser (2006) expressed the view that employing diagrams aids students in showcasing spatial reasoning abilities. In this context, integrating diagrams within geometry instruction entails teachers proficiently illustrating geometric figures visually and guiding students in identifying graphical or geometric connections present in the diagrams. According to Battista (2011), the initial introduction of geometry to primary school students occurs through motion geometry, where shapes or objects are shifted from one position to another. This motion is comprehended as distinct processes, directed by the teacher. These actions are classified as transformations, which encompass turning (rotation), sliding (translation), and mirroring (reflection). This pedagogical approach, as Battista observed, fosters the development of geometric thinking among pupils. It encourages them to explore and uncover geometric concepts according to their capacity and maturity, eventually contributing to problem-solving skills and igniting their enthusiasm for geometry

2.3 Empirical Review

This section of the literature focused on empirical research concerning the study's specific objectives. Issues concerning the effects of Problem-Based instruction on students' geometric performance, the effects of Van Hiele Phase Based teaching on students' geometric performance, were highlighted. The subsequent paragraphs throw more light on these issues

2.3.1 Effects of Problem-Based Learning on students' Geometric performance

The study by Rahmawati, Afiani and Faradita (2022) analysed how problem-based learning can be used to enhance students' performance on geometric problems. The study adopted the class action research (CAR) design. The study sampled third-grade students for the study. Generally, the study saw an improvement in students' performance in geometry. There was a significant development of student learning outcomes in affective and cognitive aspects. The findings from the study indicated that the utilization of the problem-based learning approach led to enhanced learning results among third-grade students.

Sulistyowati and colleagues (2017) carried out research comparing Problem Solving Reasoning (PSR) and Problem-Based Instruction (PBI) in relation to improving problem-solving and math communication abilities, considering Self-Regulated Learning (SRL). The study used a semi-experimental research setup and analyzed data using descriptive statistics, such as two-way multivariate analysis of variance (MANOVA) and one-way analysis of variance (ANOVA). The outcomes of the study showed that PSR was more successful than PBI in boosting problem-solving skills.

The research conducted by Simamora, Simamora and Sinaga (2017) investigated how the Problem-Based Learning Model influenced the enhancement of students' proficiency in solving geometric problems. Employing classroom action research as the research approach, the study involved a sample of 30 students from class X-6 of a Tenth Year Senior High School. The findings demonstrated that employing the problem-based learning approach led to enhancements in students' abilities to solve mathematical problems.

Jamaan, Nomida and Syahrial (2019) examined the impact of employing problem-based learning and visual-spatial intelligence on students' achievement in geometry. The research adopted a methodology involving the comparison and specific selection of distinct student groups. They utilized assessments to gauge students' proficiency in geometry and their capacity to comprehend visual-spatial information before and after the study. The collected data underwent analysis using independent samples two-way analysis of variance. The findings indicated that students instructed through problem-based learning exhibited superior performance in geometry in comparison to those taught using the discovery-learning model. Furthermore, the study revealed that students with stronger visual-spatial abilities excelled in geometry when exposed to the problem-based learning approach, as opposed to the discovery learning approach.

2.3.2 Effect of Van-Heille Theory on Student Geometric Thinking

Abdullah and Zakaria (2013) carried out a study in Malaysia to evaluate the effectiveness of Van Hiele's phase-based learning in improving students' geometric thinking abilities. Employing a quasi-experimental research design, the study involved 94 students and two teachers from a secondary school in Malaysia. The data collection period spanned six weeks, during which the students were divided into two groups: a control group and a treatment group, each comprising 47 students. Before and after the instructional intervention, both groups undertook the Van Hiele's Geometry Test (VHGT). Furthermore, ten students were selected for a focused group interview to assess their initial and final levels of geometric thinking.

To evaluate the hypotheses formulated, data analysis utilized Wilcoxon-t tests. The outcomes of the study revealed that there was no significant difference in the initial stages of geometric thinking between the two groups. Nonetheless, a notable distinction emerged in the final stages of geometric thinking between the control and treatment groups. The qualitative analysis further disclosed that during the initial stages of geometric thinking, a majority of students in both groups achieved the first Van Hiele level with a thorough understanding, had a limited grasp of level two, and lacked comprehension of level three. In the subsequent interviews, students in the control group displayed progress from level one to level two in their geometric thinking, although none reached level three. On the contrary, all students in the treatment group comprehensively mastered Van Hiele level one, with nearly all successfully advancing to level two. While only one student did not reach level three, the rest exhibited extensive and advanced comprehension.

Armah et al. (2018) conducted a study to explore the influence of van Hiele Phase-based Instruction (VHPI) on the development of geometric thinking in Ghanaian Pre-service Teachers, evaluated through van Hiele Levels. The investigation utilized a quasi-experimental pre-test post-test design, involving 150 Pre-service Teachers, evenly distributed between the experimental and control groups. All Pre-service Teachers were assessed using the Van Hiele Geometry Test (VHGT) before and after the instructional intervention. The experimental group was instructed in two-dimensional geometry using VHPI, while the control group followed conventional teaching methods. Data analysis encompassed Chi-square tests. The findings demonstrated that both sets of Pre-service Teachers displayed enhancements in their post-VHGT scores compared to their initial VHGT scores. Importantly, the Pre-service Teachers in the experimental group exhibited more substantial progress in their levels of geometric thinking in comparison to their counterparts in the control group.

Pujawan, Suryawan and Prabawati's (2020) study aimed to evaluate the effectiveness of the Van Hiele model in improving students' spatial skills within the context of the platonic solid topic. This quasi-experimental research employed a design where only a post-test was administered to a control group and an experimental group. The study was conducted over a month and centred on eighth-grade students from a junior high school in the Seririt sub-district. Using random sampling, 64 students were chosen as participants and divided into two classes. The control group received traditional teaching methods, while the experimental group was exposed to the Van Hiele learning approach. Data collection encompassed an essay test to assess spatial abilities upon completion of the study. Statistical analysis involved a one-tailed t-test. The findings of the analysis revealed that the average score of the experimental group surpassed that of the control group. Consequently, it can be inferred that the Van Hiele learning model positively impacted the enhancement of students' spatial abilities in comparison to conventional teaching methods.

Machisi (2020) conducted a study that involved a convenience sample of 186 Grade 11 students from four matched secondary schools located in the Capricorn district of the Limpopo province, South Africa. The research design utilized a sequential explanatory mixed-methods approach, which incorporated both quantitative and qualitative data collection techniques. In the quantitative phase, a non-equivalent groups quasi-experiment was undertaken. To evaluate students' proficiency in constructing geometric proofs, a Geometry Proof Test was administered prior to and following the teaching experiment. The utilization of non-parametric analysis of covariance to examine the data revealed a notable improvement in the academic performance of students attending experimental group schools, as opposed to their peers in the control group schools. In the qualitative stage, information was collected through focus group conversations and entries in students' journals. The findings indicated that students belonging to the experimental group schools held favourable perceptions of their experiences in learning geometry, while students from the control group schools exhibited unfavourable attitudes towards the teaching of Euclidean geometry and geometric proofs within their mathematics classes.

2.4 Conceptual Framework

Based on the empirical review, this study proposes relationship among van Hiele phase-based teaching, problem based learning and students' geometric thinking. Figure 1 represents the relationship.

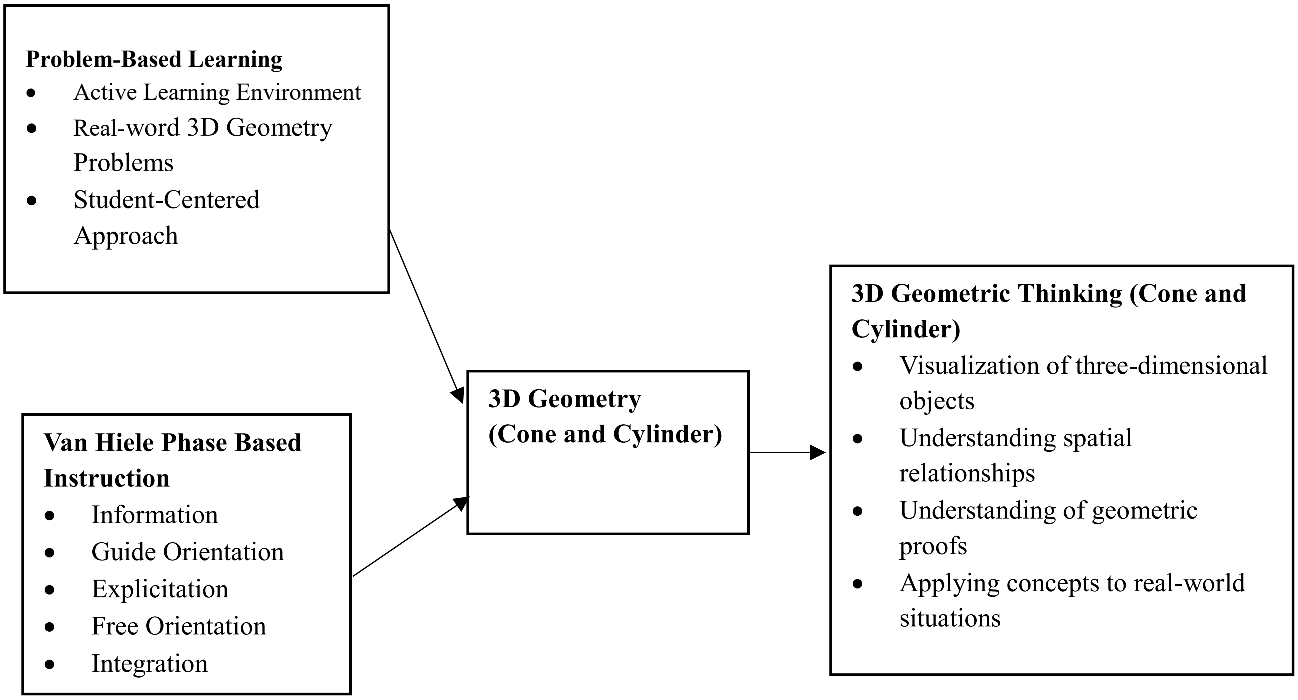

Figure 1: Relationship between Van Hielle Phase-Based Instruction and Problem-Based Learning on students' geometric thinking.

Source: *Author's own construct*

Although the Van Hielle Phase Based Instruction and Problem-Based Learning methods have been used independently to improve students' geometric thinking, this study proposes that combining the two instructional techniques on 3D geometric objects provides a powerful mix for improving students' geometric thinking. This study proposed that when these methods are combined simultaneously in the teaching of 3D geometry it will have a greater influence on students' geometric thinking than their individual constituents. This study proposed that mathematics teachers should combine the various steps and techniques embedded in the Van Hielle Phase Based Instruction and Problem-Based Learning methods in teaching geometry.

2.5 Chapter Summary

The literature review was categorized into three main sections: theoretical review, conceptual review, and empirical review. The theoretical review delved into the Van Hiele theory of geometric thinking as developed by Pierre and Dina van Hiele (1957). Likewise, the conceptual review covered topics such as the concept of geometry, geometric thinking, challenges linked to learning geometry, and approaches to teaching geometry. Lastly, the empirical review examined relevant studies aligned with the study's objectives.

CHAPTER THREE

RESEARCH METHODS

3.0 Introduction

The primary objective of this study was to enhance students' geometric thinking through the implementation of problem-based teaching and Van Hiele phase-based instruction. This chapter outlines the research methods that will be utilized in conducting the investigation. It details the systematic approach taken to collect accurate and dependable data, along with the procedures employed to analyze the data, all aimed at achieving the study's overarching goal. The chapter is structured into several subsections, including research design, study area, population, sampling method, data collection tool, data collection process, data processing and analysis, and a summary of the chapter.

3.1 Research Paradigm

The concept of a Research Paradigm, elucidated by Kivunja and Kuyini (2017), pertains to the viewpoint, mindset, theoretical approach, or collective convictions that shape the understanding and interpretation of research data. It inherently mirrors the researcher's viewpoints about the existing world and the desired world. While various research paradigms exist, this study was founded upon the positivist research paradigm. Rehman and Alharthi (2016) suggest that within positivism, it is believed that reality exists autonomously from human perception. Moreover, the scientific approach is regarded as the sole means to establish truths and objective realities (Chilisa, 2010). In this perspective, reality is not influenced by our sensory experiences and is governed by unchanging laws. Chilisa and Kawulich (2012) posit that the positivistic paradigm typically assumes a quantitative methodology The positivist methodology relies heavily on experimentation and mainly yields quantitative results. The objective of positivism is to quantify, manage, forecast, formulate laws, and assign causation. Given the objective of this study, which is to enhance the geometric thinking of senior high school students using problem-based learning and Van Hiele phase-based instruction, I find it appropriate to adopt the positivist research paradigm.

The study, therefore, adopted the positivist research paradigm. According to Park, Konge, and Artino (2020), the positivist paradigm offers approaches that can be employed to validate connections between causal and explanatory elements (independent variables) and results (dependent variables). They also pointed out that the primary objective of positivist inquiry is to establish explanatory relationships or causal connections that ultimately enable the anticipation and management of the phenomena under investigation. In alignment with this perspective, the current study, employing a quasi-experimental approach, sought to determine the impact of problem-based instruction and Van Heile's problem-based instruction on students.

3.2 Research Approach

A research approach is the plan or procedure that a researcher uses to collect, analyse and interpret data (Fraenkel, Wallen, & Hyun, 2012). This study employed the quantitative research approach. As elucidated by Fraenkel, et al. (2012), quantitative research involves the procedure of gathering, examining, and interpreting numerical data. Williams (2011) suggests that quantitative research involves the use and examination of numeric data using particular statistical methods to tackle questions. The researcher deems the quantitative approach appropriate for this study as it provides methods that can be used to test causal relationships and make predictions (Apuke, 2017).

3.3 Research Design

A crucial component of the research process is the research design, which serves as a central building block. It creates a framework that shapes the entire research undertaking (Fraenkel, Wallen, & Hyun, 2012). The research design articulates the holistic approach a researcher employs to obtain answers to research questions and validate hypotheses (Amedahe & Asamoah-Gyimah, 2015). Ultimately, the research design lays down the foundational structure of the study.

Various categories of research designs exist; nevertheless, the selection of a specific design primarily hinges on the characteristics of the research quandary at hand, the involved research inquiries, and the participant group participating in the investigation. Consistent with the objective of this study to enhance students' geometry performance through the utilization of van Hiele's geometric thinking model via problem-based learning, the quantitative research methodology was adopted, specifically the quasi-experiment design. The study adhered to Fraenkel and Wallen's (2006) pattern of pre-test and post-test. This approach is explained below:

Type of Group	**Pre-test**	**Treatment**	**Post-test**
Experimental	T_1	x	T_2
Control	T_3	C	T_4

The pre-tests (T_1 and T_3) were conducted to determine the initial entry points and compare the difference between the experimental and control groups before treatment. The post-tests (T_2 and T_4) were administered to examine the treatment effect after the experimental group received van Hiele Phase-based Instruction through problem-based learning (X) and the control group received the conventional instruction (C).

3.4 Population

A study's population refers to the specific group of interest to the researcher for the purpose of obtaining information and deriving conclusions. As outlined by Dagnachew and Sewagegn (2020), a study population represents the group from which a sample is selected and from which conclusions can be generalized. Additionally, a population is described as a collection of individuals, entities, items, or objects from which samples are gathered for measurement purposes (Kabir, 2016). The study's population included all third-year students at St. James Seminary Senior High School. Information from the academic department of the school indicates that there were a total of 680 students in the third year. The school offers three programmes these are General Science, General Arts and Business. The students are grouped into various classes within the department. There are nine General Science classes, three General Arts classes and one Business class.

3.5 Sampling Procedure

Sampling involves the procedure of choosing a subset of the population to stand for the entire population (Amedahe & Asamoah-Gyimah, 2015). The sample size of a study is the magnitude of the population included in the study (Kabir, 2016). Since the population was grouped into different classes, the cluster sampling technique was first employed to select two classes to represent the sample size. The intact class sampling was used. The researcher assigned one class as the control group and the other as the experimental group. The sample size for the study was 100 students.

3.6 Research Instrument

The main instrument for the study was a 3-dimensional Geometry Test based on the Senior High School 3 syllabus. The paper and pencil test was developed by the researcher. The instruments consisted of tests (pre-intervention and post-intervention) and a well-structured intervention exercise. Before the actual administration of the tests, pre-testing of the instruments was carried out to check whether the instruments were reliable and valid. The research instrument comprised three parts labelled sections 'A' to 'C'. Section A comprised items soliciting students' background data. The demographic variables included gender and age. Additionally, Section B contained five questions on geometric proofs on cones and cylinders. Lastly, section 'C' consisted of questions on relationships between cones and cylinders. The validity and reliability of the test were tested through content validation and a pilot test.

3.7 Pilot Testing

The pilot testing of the research instrument took place at Sunyani Senior High School. Sunyani Senior High School was chosen because it shares similar characteristics as St. James Seminary Senior High School. The pilot test was done to ascertain the item difficulty, item discrimination and item reliability for each item as well as the entire instrument. The researcher sampled 20 form two science students from Sunyani Senior High School for the pilot testing. The test was conducted on two separate days of three-day interval. The students were given two hours in each test to respond to 10 questions in the test. All students were given the same questions to respond to. In order to ensure that the students did not cheat or engage in examination malpractices, the researcher invigilated the conduct of the test and ensured that students adhered to the rules that govern the conduct of tests and examinations. This was done to ensure that cheating in the test was eliminated since cheating in a test makes the test scores unreliable (Cohen, Manion & Morrison, 2007). In addition, other factors that affect the reliability of test scores were taken into consideration. The scores for each student on the two tests were scored and correlated in order to determine the reliability of the instrument.

3.8 Reliability and Validity of Research Instrument

Errors and uncertainties in question layout and creation can be easily missed during the design of a test instrument. It's feasible to create a reliable test instrument with consistent responses, yet it might lack validity if it does not accurately measure the intended concept. In order to confirm the validity of the test, it was administered to an expert with extensive expertise in mathematics assessment. Face or content validity can be assessed through the expert evaluation of someone knowledgeable in the field (Gay, Mills & Airasian, 2009). The recommendations provided by the expert were utilized to revise the items. As outlined by Amedahe (2001), the validation pertains to the accuracy of the interpretations assigned to the assessment scores, rather than the instrument itself. When the instrument effectively measures its intended concept and the outcomes are applied for their intended use, the instrument can be considered valid.

Additionally, to ensure the reliability of the test items, the researcher employed the test-retest method in ascertaining the reliability of the research instrument. The scores from the first and second tests were correlated in determining the reliability of the test instrument. The result from the Pearson Correlation was 0.85 and a p-value of = 0.00 at 0.01 level of significance. This revealed that the test instrument has high reliability (Cohen, Manion & Morrison, 2007).

3.9 Treatment

During the study, the experimental group received the Van Hiele Phase-based Instruction (VHPI) using the problem-based learning method, while the control group followed conventional instruction. Both groups of students underwent a series of nine distinct lessons as part of the study. The topics that were treated included proofs, axioms and theorems of cones and cylinders. The lessons that were taught in the experimental group were carried out through the concepts of discussion, group work and hands-on investigations.

The students in the experimental group went through the instructional stages outlined by van Hiele (1957), which consisted of five phases. In the initial phase, known as the Information/Inquiry Phase, the students' prior knowledge regarding different topics was revisited. This process allowed the researcher to assess the students' grasp of specific geometric shapes, involving activities like making observations, posing questions, and comprehending the relevant vocabulary associated with those geometric shapes. In Phase 2 (Guided Orientation Phase), students engaged in practical activities that facilitated their acquaintance with numerous proofs and theorems related to the geometric concept. In Phase 3 (Explication Phase), students were provided with the chance to articulate in their own words the findings they had uncovered in the preceding phases. The researcher used the questioning technique to draw out students' thinking. This helped students to be familiar with relevant geometrical terminologies. In Phase 4 (Free Orientation Phase), students were tasked with solving problems and establishing connections autonomously. These exercises were crafted to offer the students challenges of a greater complexity, open-ended nature, and involving multiple potential solutions. Finally, students were taken through the fifth phase (Integration Phase) the focus of this phase was for the students to reflect on what they have learned and determine how the learning fits into the overall mathematical structure. Students were asked to review and summarize what they had learned.

3.10 Ethical Consideration

The research was conducted in strict accordance with ethical principles governing research practices. Confidentiality, anonymity, and privacy considerations were meticulously followed. Prior to the commencement of the research, the consent of students from both the control and experimental groups was obtained. First, the purpose of the study was comprehensively clarified to the participants, followed by the request for their approval through the completion of a consent form. The participants were informed that their involvement in the study was entirely voluntary and not obligatory. The gathered data was treated with utmost confidentiality, ensuring that the identities of the participants remained undisclosed throughout the research. The collected data was collectively analyzed, preventing the possibility of linking responses to specific individuals. Strict measures were implemented to securely manage and safeguard the data, preventing unauthorized access by external parties

3.11 Data Collection

Data was collected by the researcher himself through the Non-Equivalent Group Design pretest-post-test procedure. The data collection lasted for a period of three weeks. The researcher met each group for three hours every week. The first phase of the study involved the pre-intervention test for both the control and experimental groups. In the second phase, the experimental group was oriented on the Van Hiele model. In addition, they were taken through problem-based learning. They were given the opportunity to share what they had learnt. However, the control group was taken through the traditional method of teaching geometry. At the end of the period, both groups were given tests to assess the effect of the intervention.

3.12 Data Processing and Analysis

Pre-intervention and post-intervention tests for both the control and experimental groups were marked and scored. Students' test results were analysed with the aid of Statistical Packages for Social Sciences (SPSS, version 26). Specifically, Charts, descriptive statistics and independent sample Mann-Whitney U test were used to determine whether there was a significant difference between the geometric thinking levels and Test scores of students in the experimental and control groups.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.0 Overview

The aim of this research was to assess the impact of applying the van Hiele Phase-based Instruction in conjunction with problem-based learning on students' geometric thinking, particularly focusing on achieving van Hiele Level 3 understanding using three-dimensional objects. This investigation, rooted in the positivist research paradigm, was conducted using a quasi-experimental research design. Data collection comprised pre-intervention and post-intervention assessments. This chapter delves into the outcome analysis of the field-collected data, structured in two segments. The initial part discloses the research findings, while the subsequent segment engages in the discussion. The initial sections encompassed demographic details of the participants, followed by the core outcomes presented in later parts.

4.1 Demographic Characteristics of Respondents

This section presents the results on the demographic characteristics of the participants. The demographic data encompassed details like gender, age groups, and the classes of the respondents. Table 1 furnishes comprehensive information regarding the demographic features of the respondents.

Table 1: Demographic Characteristics of Respondents (n=100)

Variable	Frequency	Percentage (%)
Gender		
Male	100	100
Age-range		
5-18 years	75	75
19-21 years	18	18
22-25 years	7	7
Class of Study		
Form 3 Science 4	50	50
Form 3 Science 7	50	50

Source: Field Work, 2023

As shown in Table 1, all the students were males. This is because St James Seminary Senior High School where the study took place is a male Senior High School. Regarding the age of the respondents, most of the participants 75 (75%) were between the ages of 15 – 18 years, 18 (18%) were between the ages of 19 - 21 years, whereas only a hand few of the participants 7 (7%) were within the ages of were 22 – 25 years. The results further revealed that an equal proportion of the respondents (50%) were in Form 3 Science A and C classes.

4.2 Test of Normality

Prior to the analysis of the various research questions, a test of normality was carried out for all the pre-test and post-test results to determine whether the results are normally distributed or not. The Kolmogorov-Smirnova test of normality was used in anaysing the normality of the tests scores. Table 2 presents the results of the normality test.

Table 2: Tests of Normality

Kolmogorov-Smirnova			
	Statistic	**df**	**Sig.**
Geometric Proofs Pre-test	.175	100	.000
Relationships Pretest	.187	100	.000
Geometric thinking pretest	.131	100	.000
Geometric Proofs Post test	.082	100	.096
relationships post test	.110	100	.004
Geometric Thinking Post test	.084	100	.078

Source: Field Work, 2023

Based on the results of the Kolmogorov-Smirnov test (Table 2), the pre-test scores (Geometric Proofs Pre-test, Relationships Pretest, Geometric Thinking Pretest) significantly deviate from a normal distribution, while the post-test scores (Geometric Proofs Post-test, Relationships Post-test, Geometric Thinking Post-test) show varying levels of departure from normality, with the Relationships Post-test score exhibiting the most pronounced deviation. Additionally, the Normal Q-Q for all the tests were also examined (see Appendix D). The Normal Q-Q for all the test shows that the distributions were not closer to the straight line

4.3 Research Question One

What is the effect of problem-based learning on students' ability to perform geometric proofs on cones and cylinders?

The aim of this research inquiry was to assess how problem-based learning impacts students' capacity to carry out geometric proofs related to cones and cylinders. Prior to and after the intervention, both groups underwent pre-test and post-test assessments. These evaluations were designed to gauge the impact of problem-based learning on students' proficiency in handling geometric proofs concerning cones and cylinders.

Each pre-test and post-test consisted of five theoretical questions, with a total possible score of 50 for each test. The post-test was attended by all students in both groups

Table 3: Descriptive Statistics of Pretest and Post-tests Scores for Experimental and Control Groups (PBL on Geometric Proofs)

Pre-test					**Post-test**			
Group	**N**	**Mean**	**Median**	**Std. Deviation**	**N**	**Mean**	**median**	**Std. Deviation**
Control	50	10.02	7.0	10.95	50	17.84	18	10.58
Experimental	50	11.40	6	11.22	50	32.80	34.5	9.88

Source: Field Work, 2023

Prior to implementing the intervention, both the control and experimental groups performed a pre-intervention exam to assess their performance. The pre-test outcomes indicated that the control group had a mean score of 10.02 and a median score of 7.0. On the other hand, the experimental group exhibited mean and median scores of 11.40 and 18.0, respectively. Despite the higher median scores in the experimental group, the mean scores did not show a significant difference. The p-value (0.39) surpassed the significance level of 0.05. The pre-test findings suggested that students in both groups had comparable geometric proof abilities before the intervention. Consequently, any variations in students' performance on geometric proofs concerning cones and cylinders following the intervention could be attributed to the intervention itself.

As shown in Table 3, the descriptive statistics further show that the experimental group outperformed the control group in the posttest. Specifically, the mean and median scores of the experimental group in the post-intervention test (M = 32.80, MD = 34.5) were higher than that of the control group (M = 10.58, MD = 6).

To ascertain if there existed a noteworthy distinction between the scores of the control and experimental groups in the post-intervention test, the Mann-Whitney U test, which serves as the non-parametric counterpart to the independent sample t-test, was employed. This choice was made due to the fact that the data did not fulfill the prerequisites of parametric analysis, such as normality and homogeneity of variance. The outcomes of the test are provided in Table 4.

Table 4: Independent-Samples Mann-Whitney U Test Summary (PBL on Geometric Proofs)

Total N	**100**
Mann-Whitney U	2129.500
Wilcoxon W	3404.500
Test Statistic	2129.500
Standard Error	144.859
Standardized Test Statistic	6.071
Asymptotic Sig.(2-sided test)	.000

Source: Field Work, 2023

As depicted in Table 4, the Mann-Whitney U (U statistics) value stands at 2129.5. This value represents the sum of rank-based differences between the control and experimental groups with respect to their outcome scores. The U statistics observed in this study indicated that the ranks of the outcome scores within the experimental group tended to be higher than those within the control group. The findings demonstrated a noteworthy disparity in the scores between the experimental group (which underwent VHPB and PBL instructions) and the control group (which received conventional instruction). The p-value (0.000) was lower than the significance level of 0.05. Additionally, the effect size was determined to be 0.6071, signifying a substantial effect in accordance with Cohen’s classification.

Findings from the study indicated that prior to the implementation of the intervention, there existed no significant disparity in the scores between the control and experimental groups. This suggests that both groups of students possessed a comparable level of comprehension concerning geometric proofs related to cones and cylinders. Subsequent analysis of post-test results unveiled that students in the experimental group outperformed their counterparts in the control group. This analysis was conducted by contrasting the medians of the two groups. Notably, the experimental group exhibited a higher median (34.5) in contrast to the control group (6). This observation suggests that following the intervention, students in the experimental group, who were instructed through the PBL approach, achieved a superior level of understanding compared to their peers in the control group, who received conventional teaching. Consequently, this distinction led to improved performance within the experimental group in comparison to the control group.

Furthermore, an examination was conducted using the Mann-Whitney U test to ascertain if there existed statistical significance in the scores between the control and experimental groups. Findings from this test disclosed a substantial disparity in the Median scores of the experimental and control groups. Additionally, an evaluation of the intervention's impact was undertaken through an effect size test. The outcomes of this analysis unveiled a notable effect resulting from the intervention. This indicates that the intervention effectively enhanced students' capability to engage in geometric proofs associated with cones and cylinders.

Overall, the outcomes of this research demonstrated the efficacy of problem-based learning in augmenting students' proficiency in executing geometric proofs concerning cones and cylinders. The investigation ascertained that students who received instruction through PBL approaches exhibited notable enhancements in their performance compared to those who were taught through conventional teaching methods.

4.4 Research Question Two

What is the effect of problem-based learning on students' ability to determine the relationships between cones and cylinders?

The aim of this research inquiry was to assess the impact of problem-based learning on students' ability to discern the relationships between cones and cylinders. A preliminary assessment and a subsequent evaluation were administered to both the control and experimental groups to gauge their competence before and after the intervention. Each of these assessments encompassed 5 theoretical questions, collectively amounting to a maximum score of 50 for each test.

Table 5: Descriptive Statistics of Pre-Test Scores of Control and Experimental Groups (PBL on Relationships between cones and cylinders).

Statistics	Control	Experimental
N	50	50
Mean	9.30	10.52
Median	6.00	7.00
Std. Deviation	9.75	10.09
Skewness	1.05	.802
Range	36.00	36.00
Minimum	.00	.00
Maximum	36.00	36.00

Source: Field Work, 2023

The outcomes of the preliminary test demonstrate that the control and experimental groups exhibited nearly identical average scores, with the control group averaging at 9.30 and the experimental group at 10.52. Eleven (11) students from the control group and 9 students from the experimental group scored zero in the pre-test. The range statistic for both groups was equivalent (36), and their standard deviations were also similar, 9.75 for the control group and 10.09 for the experimental group. These findings indicate that the distribution of scores around the mean was comparable for both groups. The pre-test performance, as indicated by the mean (M) and median (MD) scores, of the experimental group (M = 10.52, MD = 7.00) did not exhibit statistically significant superiority over that of the control group (M = 9.30, MD = 6.00).

The conclusion drawn from the pre-test results is that both groups of students possessed equivalent levels of geometric proficiency before the intervention. Consequently, any disparities in their ability to discern the connections between cones and cylinders after the intervention can be attributed to the impact of the intervention itself.

The subsequent phase of addressing research question two involved evaluating the post-test scores. This evaluation is divided into three sections. The initial section provides a comprehensive overview of the post-test results. The second segment offers an analysis aimed at determining if a statistically significant disparity exists between the aptitude of students instructed through the PBL model and those instructed using the conventional teaching method in establishing the associations between cones and cylinders. This analysis employed the Mann-Whitney U test, which serves as a non-parametric counterpart to the independent sample t-test due to the data's non-conformity with the prerequisites for parametric analysis, such as normality and uniform variance.

Table 6: Descriptive Information of Post-Test Scores for Control and Experimental Groups (PBL on Relationships between cones and cylinders)

Statistics	Control	Experimental
N	50	50
Mean	20.74	30.00
Median	20.00	35.00
Std. Deviation	11.80	13.34
Skewness	.148	-.116
Range	44.00	44.00
Minimum	.00	6.00
Maximum	44.00	50.00

Source: Field Work, 2023

The outcomes of the post-intervention assessment (Table 6) disclosed that the mean score for the experimental group (30.0) surpassed the mean score of the control group (20.74). This difference in mean scores between the two groups amounted to 9.26. This indicates that the average score of the experimental group exceeded that of the control group by 9.26 points. Within the experimental group, the maximum score was 50, while the lowest was 6. This implies that three students within the experimental group achieved a perfect score of 50 points on the post-test. In comparison, the minimum score for the control group was zero (0), while for the experimental group, it was six (6). The range of scores for both the experimental and control groups remained consistent, at 44. This reveals that the discrepancy between the highest and lowest scores was identical in both groups. The experimental group displayed a standard deviation of 13.34, whereas the control group had a standard deviation of 11.80. The standard deviation measure suggests that the deviation of scores from the experimental group's mean score was slightly wider than that of the control group.

The Mann-Whitney U test was conducted to determine whether there was a statistical and significant difference between the mean and median of the control and experimental groups.

Table 7: Independent-Samples Mann-Whitney U Test Summary (PBL on Relationships between cones and cylinders)

Total N	**100**
Mann-Whitney U	1711.50
Wilcoxon W	2986.50
Test Statistic	1711.50
Standard Error	144.90
Standardized Test Statistic	3.185
Asymptotic Sig.(2-sided test)	.001

Source: Field Work, 2023

As depicted in Table 7, the Mann-Whitney U (U statistics) is recorded as 1711.50. This figure denotes the cumulative rank-based disparity between the control and experimental groups in relation to their outcome scores. The U statistics extracted from the study unveiled that the ranks of outcome scores within the experimental group tended to surpass those within the control group. The findings indicated a substantial disparity in scores between the experimental group (receiving PBL instructions) and the control group (exposed to conventional instruction) concerning the determination of the relationship between cones and cylinders. The p-value (0.001) was lower than the significance threshold of 0.05. Notably, the effect size equated to 0.3185, denoting a moderate effect size as categorized by Cohen's classification.

The findings of the present study unveiled that prior to the implementation of the intervention, there existed no statistically significant distinction between the scores of the control and experimental groups. This suggests that students in both groups possessed a comparable level of understanding regarding the determination of relationships between cones and cylinders. This similarity was reflected in the mean and median scores of both groups. Upon conducting the post-test, the results indicated that students in the experimental group outperformed their peers in the control group. This was evaluated through a comparison of the medians of these two groups. The experimental group exhibited a higher median score in comparison to the control group. Consequently, the experimental group, taught using the PBL approach, showcased an enhanced understanding compared to their counterparts in the control group, instructed using conventional teaching methods. As a result, the experimental group's performance surpassed that of the control group.

Furthermore, the Mann-Whitney U test was executed to ascertain whether there existed statistical significance in the scores of the control and experimental groups. The outcomes of this test disclosed a noteworthy distinction between the median scores of the experimental and control groups. Furthermore, an assessment of effect size was carried out to gauge the impact of the intervention. The findings from this analysis indicated a moderate effect resulting from the intervention. This signifies that the intervention proved to be efficacious in enhancing students' capacity to determine the relationships between cones and cylinders.

4.5 Research Hypothesis

H_0: There is no statistically significant difference among SHS 3 students' performance in level three of van Hiele's levels of geometric thinking, regarding the methods by which they are taught (conventional instruction versus VHPI through problem-based learning).

H_1: There is a statistically significant difference among SHS 3 students' performance in level three of van Hiele's levels of geometric thinking, regarding the methods by which they are taught (conventional instruction versus VHPI).

This research hypothesis aimed to determine whether there was a statistically significant difference between the performance of students taught with conventional instruction and VHPI through PBL in level three of van Hiele's levels of geometric thinking. A pre-test and post-test were conducted to determine the performance of both the control and experimental groups before and after the intervention. Both tests consisted of 10 theory questions each which were scored at a total of 100 each. The first part of this section presents the descriptive statistics of the post-test scores.

Table 8: Descriptive Information of Pre-Test Scores for Control and Experimental Groups (VHPBI on Geometric Thinking)

Statistics	**Control**	**Experimental**
N	50	50
Mean	19.32	21.92
Median	18.00	19.00
Std. Deviation	17.51	19.33
Skewness	1.044	1.063
Range	69.00	78.00
Minimum	0.00	0.00
Maximum	69.00	78.00

Source: Field Work, 2023

The findings presented in Table 8 which displays the pre-test scores, indicate that the control and experimental groups achieved nearly equal average marks, with 19.32 for the control group and 21.92 for the experimental group. The difference between the mean scores of these two groups was 2.6, implying a minimal discrepancy in means. Additionally, the variance in the median score was merely one (1). Notably, both groups had a minimum score of zero (0). The range statistic exhibited a value of 69.0 for the control group and 78.0 for the experimental group. Moreover, the standard deviations for both groups were approximately the same: 17.51 for the control group and 19.33 for the experimental group. These outcomes suggest that the dispersion of marks from the mean scores of both groups was relatively similar. Consequently, the performance (pre-test score) of the experimental group (M = 21.92, MD = 19) did not exhibit a statistically significant superiority over the performance (pre-test score) of the control group (M = 19.32, MD = 18). The inference drawn from the pre-test results suggests that students in both groups possessed equivalent levels of geometric thinking prior to the intervention. As a consequence, any disparities in students' geometric thinking subsequent to the intervention can be attributed to the impact of the intervention itself.

The subsequent phase of addressing the research hypothesis involved examining the outcomes of the post-test scores. This presentation is divided into three sections. The initial part offers descriptive details regarding the post-test results. The subsequent part presents the analysis aimed at assessing whether a statistically significant distinction existed between the capabilities of students taught using the VHPBI model and those taught using the conventional teaching method to ascertain the relationships between cones and cylinders. This assessment was carried out through the utilization of the Mann-Whitney U test due to the non-normal distribution of the post-test results. The post-intervention test did not satisfy the prerequisites for parametric analysis, including normality and homogeneity of variance.

Table 9: Post-test Score Descriptives (VHPBI on Geometric Thinking)

Statistics	**Control**	**Experimental**
N	50	50
Mean	38.58	62.54
Median	39.5000	68.5000
Std. Deviation	19.11	19.59
Skewness	-.026	-.223
Range	76.00	76.00
Minimum	1.00	20.00
Maximum	77.00	96.00

Source: Field Work, 2023

The outcomes of the post-intervention test unveiled that the average score for the experimental group (62.54) exceeded the average score of the control group (38.58). The disparity between the mean scores of the two groups equated to 23.96. This indicates that the mean score of the experimental group outperformed that of the control group by a margin of 23.96 marks. The highest score attained by the experimental group was 96, while the lowest score was 20. Furthermore, the lowest score of the experimental group exceeded that of the control group (20 compared to 1). The control group achieved a minimum score of one (1), whereas the experimental group attained a minimum score of 20. The range of scores for both the experimental and control groups was identical. For both groups, the range between the highest and lowest scores remained constant at 76. This illustrates a broader difference between the highest and lowest scores within both groups. The experimental group exhibited a standard deviation of 19.59, while the control group had a standard deviation of 19.11. Based on the standard deviation measure, it can be inferred that the dispersion of scores from the mean score of the experimental group was slightly larger than that of the control group.

Table 10: Independent-Samples Mann-Whitney U Test Summary (VHPBI on Geometric Thinking)

Statistics	
Total N	100
Mann-Whitney U	2003.50
Wilcoxon W	3278.50
Test Statistic	2003.50
Standard Error	145.02
Standardized Test Statistic	5.196
Asymptotic Sig.(2-sided test)	.000

Source: Field Work, 2023

As indicated in Table 10, the Mann-Whitney U statistic was recorded as 2003.50. This statistic denotes the sum of rank-based differences between the control and experimental groups in relation to their outcome scores. The U statistics extracted from the study unveiled that the ranks of the outcome scores within the experimental group tended to be higher than those within the control group. The outcomes demonstrated a substantial disparity in the scores of the experimental group (VHPBI through PBL) in comparison to the control group (receiving conventional instruction) in terms of geometric thinking. The calculated p-value (0.000) was below the predetermined significance level of 0.05. The assessment of effect size yielded a substantial value of 0.519.

The findings derived from the current study disclosed that prior to the implementation of the intervention, there existed no statistically significant distinction in the scores of the control and experimental groups. This indicates that students in both cohorts demonstrated nearly identical levels of geometric thinking. This fact was evident through the examination of the mean and median scores of both groups. However, subsequent to the post-test, outcomes indicated that students within the experimental group exhibited superior performance compared to their counterparts in the control group. This contrast was assessed by comparing the medians of the two groups. Notably, the experimental group showcased a higher median score than the control group. This implies that following the intervention, students in the experimental group, who were instructed using the VHPBI through the PBL approach, acquired a more profound understanding compared to their peers in the control group who were taught using the conventional teaching method. Consequently, the experimental group exhibited superior performance in relation to the control group.

Furthermore, the Mann-Whitney U test was executed to ascertain the presence of statistical significance between the scores of the control and experimental groups. Outcomes from this test indicated a substantial disparity between the Median scores of the experimental and control groups. Additionally, an assessment of effect size was carried out to gauge the impact of the intervention. The findings unveiled a moderate effect of the intervention. This signifies that the intervention effectively contributed to the enhancement of students' geometric thinking.

4.5 Discussion

This section delves into the outcomes of the research as elucidated in the preceding sections. The ensuing discourse is structured around the subsequent thematic points:

i. effects of problem-based learning on students' ability to perform geometric proofs on cones and cylinders.
ii. effects of problem-based learning on students' ability to determine the relationships between cones and cylinders.

iii. effect of Van Hiele theory-based instruction on students' geometric thinking on three-dimensional objects through problem-based learning

4.5.1 Effects of Problem-Based Learning on Students' Ability to Perform Geometric Proofs on Cones and Cylinders

The findings of the study revealed that generally, Problem-Based Learning is effective in enhancing students' ability to perform geometric proofs on cones and cylinders. Specifically, the study found that students who received instruction based on the Problem-Based Learning approach performed better than students who received the conventional instruction approach toward the teaching of geometric proofs on cones and cylinders.

The results from this study are in agreement with the study by Rahmawatim et al. (2022) who analysed how problem-based learning can be used to enhance students' performance on geometric problems and discovered that problem-based learning. Their study saw an improvement in the performance of students after they received instruction through the problem-based learning approach. The study saw a significant development of student learning outcomes in affective and cognitive aspects after they were taught through the problem-based learning approach. The findings of this study also agree with that of the study by Simamora, et al. (2017) who examined the effect Problem-Based Learning Model in improving students' performance in solving geometric problems. The findings of Simamora, et al (2017) revealed that problem-based learning improved the performance of students in solving geometric problems in particular and mathematics in general.

The findings of this study also corroborate the findings of Jamaan, et al. (2019) who investigated the effect of problem-based learning model and visual-spatial intelligence on students' geometry achievement by comparing problem-based learning with discovery learning model. The findings of Jamaan, et al. (2019) revealed that found the geometry learning outcomes of students learning with the PBL model were higher than those learning with the DL (discovery learning) model.

In contrast, the findings of this study are at odds with those of Sulistyowati et al. (2017), who conducted a comparative study involving Problem Solving Reasoning (PSR) and Problem-Based Instruction (PBI) to evaluate problem-solving and mathematical communication proficiencies through the lens of Self-Regulated Learning (SRL). Sulistyowati et al.'s (2017) findings suggested that PSR outperforms PBI in terms of enhancing problem-solving abilities.

4.5.2 Effects of Problem-Based Learning on Students' Ability to Determine the Relationships Between Cones and Cylinders

The findings from the study reveal that students who received Problem-Based instruction outperformed their counterparts who received conventional instruction. The study found that the median score of the experimental group in the post-test results was higher than the median score of the control group.

The findings of the current study revealed that problem-based learning if effective in enhancing students' ability to determine the relationship between cones and cylinders. The study found that when students are taught using the PBL approaches their performance improved significantly than those taught using the conventional teaching method.

These results align with the outcomes of the study conducted by Mensah, Atteh, Boadi, and Assan-Donkoh (2022), which examined the influence of the Problem-based Learning (PBL) approach in teaching mathematical problem-solving related to circles within the context of geometry. Their research distinctly demonstrated a significant enhancement in students' abilities to solve mathematical problems involving circles within geometry when they were taught using the problem-based learning approach.

The results of this study correspond with those of Jamaan, Musnir, and Syahrial (2020), who investigated the impact of the problem-based learning model on students' achievement in geometry while accounting for their initial mathematics abilities. Their findings indicated that students who engaged in the problem-based learning model achieved higher outcomes in geometry learning compared to those who used a scientific learning approach, even after accounting for their initial mathematics abilities.

The results of this study align with those of Effiom and Abdullahi (2021), who examined the impacts of the Problem-Based Learning (PBL) strategy on the Mathematical Reasoning (MR) of senior secondary students in the context of geometry. Their research identified notable disparities in the mean scores of Mathematical Reasoning (MR) among students who were instructed in geometry using the Problem-Based Learning (PBL) approach.

4.5.3 Effect of Van Hiele Theory-Based Instruction on Students' Geometric Thinking on Three-Dimensional Objects Through Problem-Based Learning

The findings of the current study revealed that the VHPBI through PBL approaches is effective in enhancing students' geometric Thinking on Three-Dimensional Objects. The study found that when students are taught using the VHPBI through PBL approaches their performance improved significantly than those taught using the conventional teaching method.

The results obtained in this study are in line with the outcomes of Abdullah and Zakaria’s (2013) research, where they evaluated the efficacy of Van Hiele’s phase-based learning in terms of students’ levels of geometric thinking. The findings of their study demonstrated a notable distinction in the final levels of geometric thinking between students taught through the VHPBI and those instructed through the conventional teaching method. Particularly, the research highlighted that students who received VHPBI-based instruction exhibited superior performance compared to those who were taught using the conventional teaching approach.

The outcomes of this study are also in alignment with the research by Armah et al. (2018), which delved into the impact of van Hiele Phase-based Instruction (VHPI) on the geometric thinking of Ghanaian Pre-service Teachers, considering the van Hiele Levels. Their investigation unveiled that Pre-service Teachers in the experimental group, who received VHPI-based instruction in two-dimensional geometry, achieved enhanced levels of geometric thinking compared to their counterparts in the control group who were taught using conventional methods. Similarly, the findings of this study correspond with the research conducted by Pujawan et al. (2020), which aimed to assess whether the Van Hiele model could offer enhanced support for students’ spatial abilities in platonic solid topics. Pujawan et al. (2020) reported that the mean score of the experimental class was higher than that of the control class, indicating that the Van Hiele learning model had a positive impact on improving students’ spatial abilities in comparison to conventional learning approaches.

The results of the present study also align with the research by Machisi and Feza (2021), which examined the impact of instruction based on the Van Hiele theory on students' competencies in geometric proofs in South Africa. Their study identified a statistically significant distinction in students' performance between the treatment and control groups. Moreover, the study revealed that the treatment group achieved higher post-test scores compared to the control group. Machisi and Feza (2021) concluded that instruction based on the Van Hiele theory is more effective in teaching non-routine geometric proofs than conventional teaching methods. These findings are also in line with the outcomes of Machisi (2020), who investigated the effectiveness of VHPBI in enhancing students' capacity to solve problems related to Euclidean geometry and geometric proofs. Machisi (2020) discovered that the experimental group's performance was notably superior to that of their counterparts in the control group.

4.6 Chapter Summary

The study examined the effect of the use of van Hiele Phase-based Instruction through problem-based learning on students' geometric thinking in terms of the van Hiele Level 3 using three-dimensional objects. The study revealed that PBL is effective in improving students' ability to perform geometric proofs on cones and cylinders. The study further revealed that PBL improves students' performance in determining the relationship between cones and cylinders. Lastly, the study found that students' geometric thinking improved significantly after they received instruction based on the VHPBL model through PBL.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.0 Overview

This chapter provides an overview of the research, outlines the conclusions drawn from the study, and encompasses suggestions and recommendations for future research. The suggestions and recommendations for further studies have been formulated in accordance with the study's findings.

5.1 Summary of the Study

The study aimed at improving students' geometric thinking by using problem-based learning and Van Heile phase-based instruction at St. James Seminary Senior High School in the Sunyani Municipality of the Bono Region in Ghana. The study was guided by three objectives which were transformed into three hypotheses. The study which was grounded in the positivist paradigm employed a quantitative research approach and a quasi-experimental design in analysing the data collected. The population of this study comprised all Form three students of St James Seminary Senior High School. The intact class sampling was used in selecting two classes for the study. The data collection instrument comprised of pre-intervention and post-intervention tests that were conducted before and after the intervention respectively.

5.2 Key Findings

The study revealed the following findings:

1. The findings of this research demonstrated the efficacy of problem-based learning (PBL) in improving students' proficiency in geometric proofs. The investigation unveiled that students who were exposed to PBL instruction

exhibited superior performance compared to those who were taught through conventional methods. Furthermore, the study revealed a notable and statistically significant distinction in the achievements of the experimental group in contrast to the control group.

2. The findings from this study unveiled that problem-based learning (PBL) enhances students' proficiency in discerning connections between cones and cylinders. The analysis highlighted that students instructed via the PBL approach achieved higher mean and median scores in comparison to those exposed to conventional teaching methods. Additionally, the study identified a statistically substantial discrepancy in the performance of students instructed with PBL and those taught through conventional methods.
3. The findings from this study illuminated that the Van Hiele Phase-based Instruction (VHPBI) integrated with problem-based learning (PBL) proves effective in elevating students' geometric cognition. The investigation underscored that learners who were educated in geometry through the VHPBI-PBL fusion achieved superior mean and median scores compared to peers who underwent traditional instructional methods. The findings consistently indicated a statistically noteworthy contrast in the academic achievements of students who underwent VHPBI-PBL and those who underwent conventional instruction.

5.3 Conclusions

Based on the findings of the study, the following conclusions are drawn the subsequent conclusions can be deduced. Firstly, it can be inferred that problem-based learning (PBL) contributes to the enhancement of students' aptitude for conducting geometric proofs. This signifies that when high school students engage with geometry through the PBL approach, their performance surpasses that of peers taught using conventional pedagogical methods.

Secondly, the incorporation of problem-based learning (PBL) into geometry instruction facilitates the enhancement of students' comprehension skills in discerning the connections between cones and cylinders. This signifies that notable progress in students' achievements is attributed to their capability to identify and comprehend the correlations inherent in cones and cylinders. Thirdly, the application of Van Hiele Phase-Based Instruction (VHPBI) coupled with problem-based learning (PBL) in geometry education elevates students' geometric thinking abilities.

5.4 Recommendations

Based on the findings of this investigation, the following suggestions are put forth for implementation by the Ghana Education Service and other concerned entities.

1. Teachers should be introduced to VHPBI and PBL as instructional methods for teaching geometry. This will contribute to the enhancement of mathematics teachers' instructional techniques and strategies, ultimately resulting in the advancement of students' geometric reasoning abilities.
2. The van Hiele theory is an important approach used in mathematics education in countries like the USA, Britain, Netherlands, and Russia. Based on this, it is suggested that Ghana's senior high school (SHS) mathematics curriculum should also incorporate the van Hiele theory. This alignment would help students improve their mathematical learning and achieve better outcomes.

5.5 Suggestions for Further Studies

The following are suggestions for further research:

1. Further studies can examine the influence of gender on the effectiveness of PBL and VHPBI on students' geometric thinking.
2. This study was limited to only two Form Three classes from one senior high school. Further studies could be replicated for a much larger sample for better generalization.

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APPENDICES

APPENDIX A

INTRODUCTORY LETTER

AKENTEN APPIAH-MENKA UNIVERSITY
of Skills Training and Entrepreneurial Development

ACADEMIC AFFAIRS DIRECTORATE

Our Ref.: 8210160004

August 17, 2023

TO WHOM IT MAY CONCERN

LETTER OF INTRODUCTION
MR. TECHIE AGYEMANG

We write to introduce Mr. Techie Agyemang (**Student No: 8210160004**) and to confirm that he is a **bona fide** student pursuing Master of Philosophy (Mathematics Education) programme at Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development; a new public tertiary University established by Act 1026, 2020 (the erstwhile College of Technology Education, Kumasi of the University of Education, Winneba).

Mr. Agyemang is currently in the second year of his programme. He was admitted in August 2021 and he is expected to complete the programme by December 2023.

We should appreciate any courtesies that could be extended to him, please.

CHARLES B. CAMPION, ACIS, PhD.
Director, Academic Affairs

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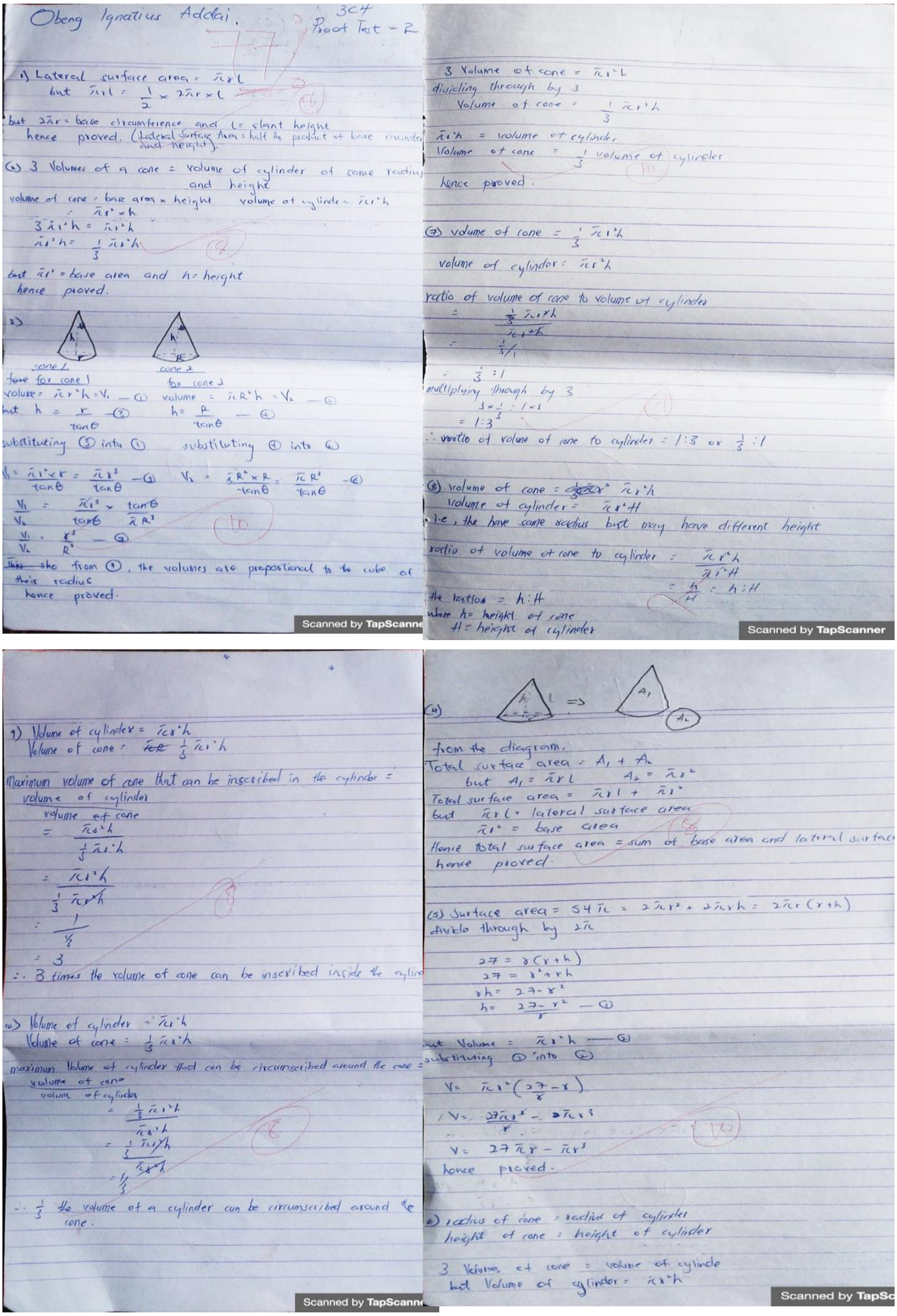
APPENDIX B

SAMPLE OF STUDENTS TEST SCORES



Obeng Ignatius Addai 3C4 Proof Test - 2 77

1) Lateral surface area $= \pi r l$
but $\pi r l = \frac{1}{2} \times 2\pi r \times l$

but $2\pi r$ = base circumference and l = slant height
hence proved. (Lateral Surface Area = half the product of base circumference and height).

(2) 3 Volumes of a cone = volume of cylinder of same radius and height
volume of cone = base area × height   volume of cylinder = $\pi r^2 h$
$= \pi r^2 \times h$
$3\pi r^2 h = \pi r^2 h$
$\pi r^2 h = \frac{1}{3}\pi r^2 h$
but πr^2 = base area and h = height
hence proved.

3) cone 1   cone 2
for cone 1: volume $= \pi r^2 h = V_1$ — ①   for cone 2: volume $= \pi R^2 h = V_2$ — ②
but $h = \frac{r}{\tan\theta}$ — ③ $h = \frac{R}{\tan\theta}$ — ④
substituting ③ into ①   substituting ④ into ②
$V_1 = \frac{\pi r^2 \times r}{\tan\theta} = \frac{\pi r^3}{\tan\theta}$ — ⑤ $V_2 = \frac{\pi R^2 \times R}{\tan\theta} = \frac{\pi R^3}{\tan\theta}$ — ⑥
$\frac{V_1}{V_2} = \frac{\pi r^3}{\tan\theta} \times \frac{\tan\theta}{\pi R^3}$
$\frac{V_1}{V_2} = \frac{r^3}{R^3}$ — ⑦
from ⑦, the volumes are proportional to the cube of their radius
hence proved.

3 Volume of cone $= \pi r^2 h$
dividing through by 3
Volume of cone $= \frac{1}{3}\pi r^2 h$
$\pi r^2 h$ = volume of cylinder
Volume of cone $= \frac{1}{3}$ volume of cylinder
hence proved.

(7) volume of cone $= \frac{1}{3}\pi r^2 h$
volume of cylinder $= \pi r^2 h$
ratio of volume of cone to volume of cylinder
$= \frac{\frac{1}{3}\pi r^2 h}{\pi r^2 h}$
$= \frac{1}{3}/1$
$= \frac{1}{3} : 1$
multiplying through by 3
$3 \times \frac{1}{3} : 1 \times 3$
$= 1:3$
∴ ratio of volume of cone to cylinder $= 1:3$ or $\frac{1}{3}:1$

(8) volume of cone $= \pi r^2 h$
volume of cylinder $= \pi r^2 H$
i.e. the have same radius but may have different height
ratio of volume of cone to cylinder $= \frac{\pi r^2 h}{\pi r^2 H} = \frac{h}{H} = h:H$
the ratios $= h:H$
where h = height of cone
H = height of cylinder

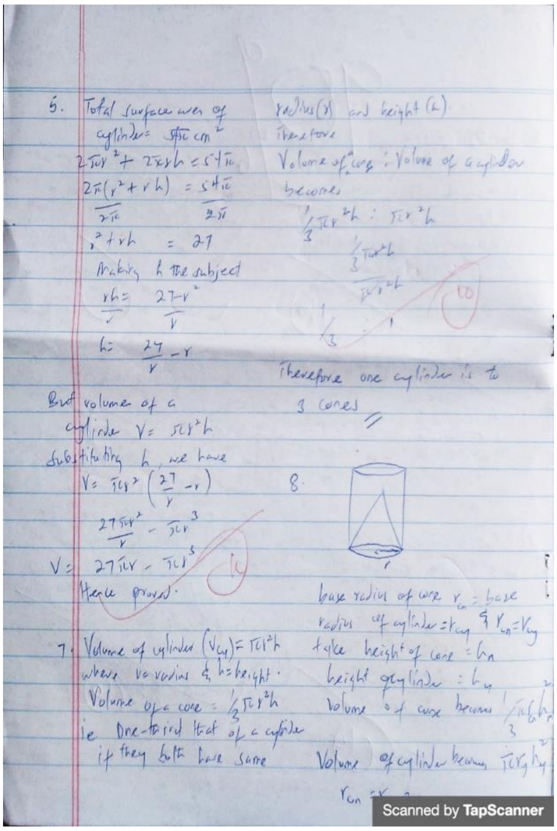
9) Volume of cylinder $= \pi r^2 h$
Volume of cone $= \frac{1}{3}\pi r^2 h$
Maximum volume of cone that can be inscribed in the cylinder = volume of cylinder / volume of cone
$= \frac{\pi r^2 h}{\frac{1}{3}\pi r^2 h}$
$= \frac{1}{1/3}$
$= 3$
∴ 3 times the volume of cone can be inscribed inside the cylinder

10) Volume of cylinder $= \pi r^2 h$
Volume of cone $= \frac{1}{3}\pi r^2 h$
maximum volume of cylinder that can be circumscribed around the cone = volume of cone / volume of cylinder
$= \frac{\frac{1}{3}\pi r^2 h}{\pi r^2 h}$
$= \frac{1}{3}$
∴ $\frac{1}{3}$ the volume of a cylinder can be circumscribed around the cone.

(4) A_1 A_2
from the diagram,
Total surface area $= A_1 + A_2$
but $A_1 = \pi r l$ $A_2 = \pi r^2$
Total surface area $= \pi r l + \pi r^2$
but $\pi r l$ = lateral surface area
πr^2 = base area
Hence total surface area = sum of base area and lateral surface
hence proved.

(5) Surface area $= 54\pi = 2\pi r^2 + 2\pi r h = 2\pi r(r+h)$
divide through by 2π
$27 = r(r+h)$
$27 = r^2 + rh$
$rh = 27 - r^2$
$h = \frac{27 - r^2}{r}$ — ①
but Volume $= \pi r^2 h$ — ②
substituting ① into ②
$V = \pi r^2\left(\frac{27 - r}{r}\right)$
$V = \frac{27\pi r^2}{r} - \pi r^3$
$V = 27\pi r - \pi r^3$
hence proved.

6) radius of cone = radius of cylinder
height of cone = height of cylinder
3 Volume of cone = volume of cylinder
but Volume of cylinder $= \pi r^2 h$


5. Total surface area of cylinder = 54π cm^2
$2\pi r^2 + 2\pi rh = 54\pi$
$2\pi(r^2 + rh) = 54\pi$
$r^2 + rh = 27$
Making h the subject
$rh = 27 - r^2$
$h = \frac{27}{r} - r$
But volume of a cylinder $V = \pi r^2 h$
Substituting h, we have
$V = \pi r^2\left(\frac{27}{r} - r\right)$
$V = 27\pi r - \pi r^3$
Hence proved.
7. Volume of cylinder $(V_{cy}) = \pi r^2 h$ where r = radius & h = height.
Volume of a cone $= \frac{1}{3}\pi r^2 h$
ie One-third that of a cylinder if they both have same radius (r) and height (h)
Therefore
Volume of cone : Volume of a cylinder becomes
$\frac{1}{3}\pi r^2 h : \pi r^2 h$
$\frac{1}{3} : 1$
Therefore one cylinder is to 3 cones
8. base radius of cone r_{cn} = base radius of cylinder $= r_{cy}$ & $r_{cn} = r_{cy}$
take height of cone $= h_n$
height of cylinder $= h_{cy}$
Volume of cone becomes $\frac{1}{3}\pi r^2 h$
Volume of cylinder becomes $\pi r_{cy}^2 h_{cy}$
$r_{cn} = r_{cy}$
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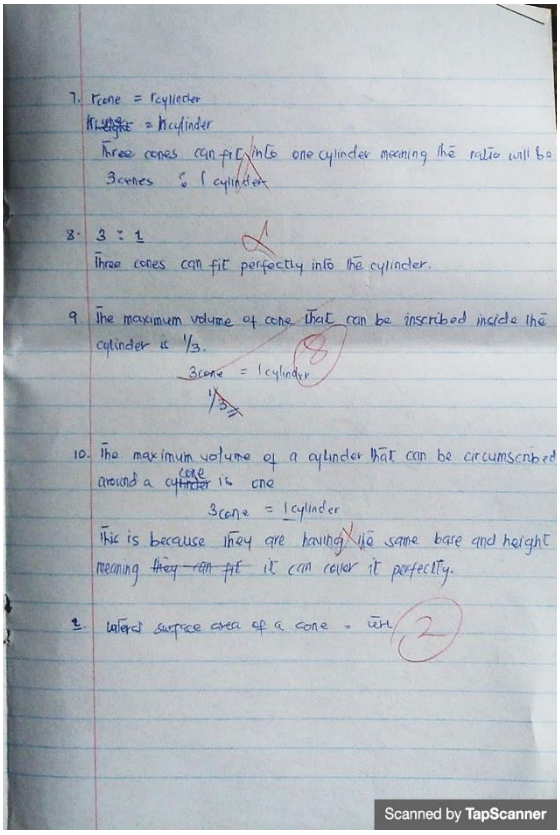
7. $r_{cone} = r_{cylinder}$
$h_{cone} = h_{cylinder}$
Three cones can fit into one cylinder meaning the ratio will be
3 cones : 1 cylinder
8. 3 : 1
Three cones can fit perfectly into the cylinder.
9. The maximum volume of cone that can be inscribed inside the cylinder is 1/3.
3 cone = 1 cylinder
1/3
10. The maximum volume of a cylinder that can be circumscribed around a cone is one
3 cone = 1 cylinder
This is because they are having the same base and height meaning ~~they can fit~~ it can cover it perfectly.
2. lateral surface area of a cone = πrl
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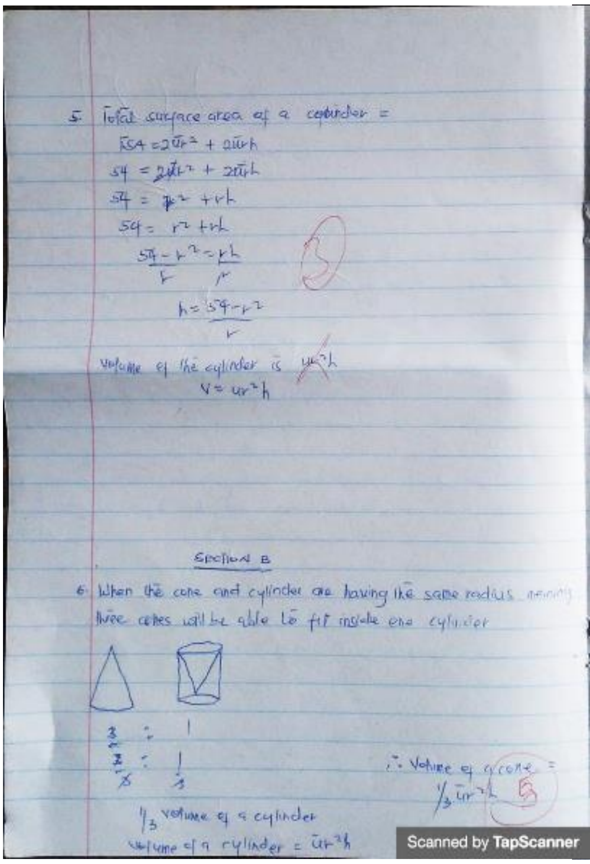
5. Total surface area of a cylinder =
$54 = 2\pi r^2 + 2\pi rh$
$54 = 2\pi r^2 + 2\pi rh$
$54 = r^2 + rh$
$54 = r^2 + rh$
$\frac{54 - r^2}{r} = \frac{rh}{r}$
$h = \frac{54 - r^2}{r}$
Volume of the cylinder is $\pi r^2 h$
$V = \pi r^2 h$
SECTION B
6. When the cone and cylinder are having the same radius meaning three cones will be able to fit inside one cylinder
$\frac{3}{3} : \frac{1}{3}$
$\frac{1}{3}$ volume of a cylinder
Volume of a cylinder $= \pi r^2 h$
∴ Volume of a cone $= \frac{1}{3}\pi r^2 h$
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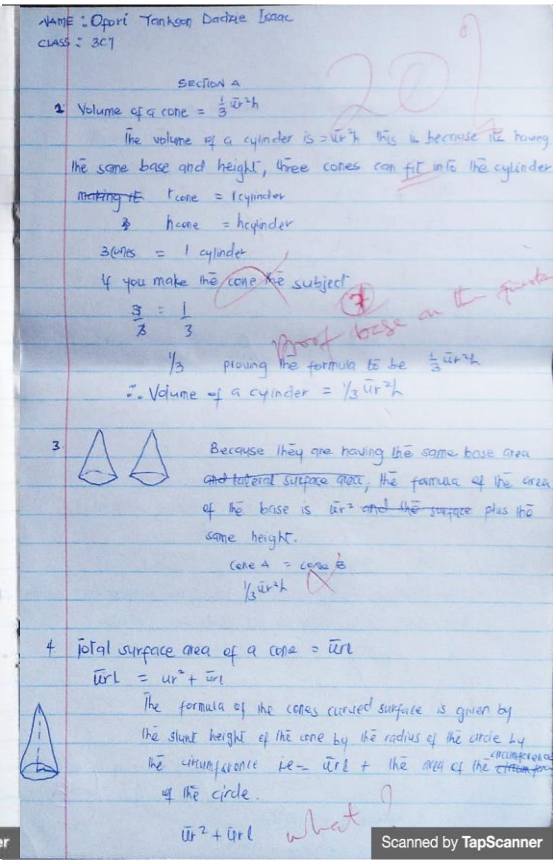
NAME: Opori Tankson Dadzie Isaac
CLASS: 3C1
SECTION A
1. Volume of a cone $= \frac{1}{3}\pi r^2 h$
The volume of a cylinder is $= \pi r^2 h$ this is because it's having the same base and height, three cones can fit into the cylinder
~~making it~~ $r_{cone} = r_{cylinder}$
$h_{cone} = h_{cylinder}$
3 cones = 1 cylinder
If you make the cone the subject
$\frac{3}{3} = \frac{1}{3}$
1/3 proving the formula to be $\frac{1}{3}\pi r^2 h$
∴ Volume of a cylinder $= \frac{1}{3}\pi r^2 h$
3. Because they are having the same base area ~~and lateral surface area~~, the formula of the area of the base is πr^2 ~~and the surface~~ plus the same height.
Cone A = Cone B
$\frac{1}{3}\pi r^2 h$
4. Total surface area of a cone $= \pi rl$
$\pi rl = \pi r^2 + \pi rl$
The formula of the cones curved surface is given by the slant height of the cone by the radius of the circle by the circumference ie πrl + the area of the circumference of the circle.
$\pi r^2 + \pi rl$
what?
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OHENEBA EMMANUEL OWUSU

307. E-MATHS POST TEST.

90/9

1) lateral surface area of cone $= \pi rl$ —(1)
where r = radius and l = slant height.
The base circumference $= 2\pi r$
$\frac{1}{2} \times$ product of slant height & base circumference $= \frac{1}{2} \times l \times 2\pi r$
$= \pi rl$ (10)
This is equal to eqn (1)
$\therefore \pi rl = \frac{1}{2} \times l \times 2\pi r$
$\pi rl = \pi rl$
Hence lateral surface area of cone is equal to half the product of slant height and its base circumference.

2) Volume of cylinder $= \pi r^2 h$
Since three cones can fit into a cylinder, if they have same radius & height, volume of one cone becomes one-third that of a cylinder $\therefore \frac{1}{3}\pi r^2 h$
where πr^2 = base area of cone and h = height. (10)

4) base area of a cone $= \pi r^2$ and lateral surface area $= \pi rl$
Total surface area of cone is given by $= \pi r^2 + \pi rl$ which is the sum of its base area and its lateral surface area.
Hence proved (10)

6) Volume of cone $= V_{cn}$ and Volume of cylinder $= V_{cy}$ $r_{cone} = r_{cylinder}$ and $h_{cone} = h_{radius}$ where r = radius, h = height.
Volume of a cylinder $= \pi r^2 h$ from analysis three cones can fit into one cylinder if they have the same radius and height.
$\therefore$ the volume of one cone becomes one-third the volume of a cylinder which is given by $\frac{1}{3}\pi r^2 h$.
Hence proved (10)

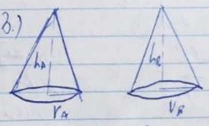
Volume of cone : Volume of cylinder becomes
$\frac{1}{3}\pi r h_n^2 : \pi r h_y^2$
$\frac{1}{3} h_n^2 : h_y^2$
with the heights not being same
$\frac{1}{3}$ the height of cone squared : the height squared of a cylinder. (10)

9. Cylinder radius r & height h
$V_{cylinder} = \pi r^2 h$
Volume of cone that can fit becomes $\frac{1}{3}\pi r^2 h$ since three cones can fit into a cylinder if they have the same radius and height. (10)

10. Cone radius $= r$ and height $= h$
~~The size of cone of~~

Only one cylinder can be circumscribed around three cones perfectly if they have same radius and height. The $V_{cone} = \frac{1}{3}\pi r^2 h$
$\therefore$ The maximum volume of the cylinder become $V_{cylinder} = \pi r^2 h$. (10)

3.)



Cone A Cone B
~~Take $r_A = r_B$~~ Take $h_A = h_B = h$
$V_A = \frac{1}{3}\pi r_A^2 h_A$ $V_B = \frac{1}{3}\pi r_B^2 h_B$
Making h_A & h_B the subjects
$V_A = \frac{1}{3}\pi r_A^2 h_1$ —(1) $V_B = \frac{1}{3}\pi r_B^2 h$ —(2)
$\frac{3V_A}{\pi r_A^2} = h_A$ —(3) $\frac{3V_B}{\pi r_B^2} = h_B$ —(4)
Since $h_A = h_B$

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APPENDIX C

GEOMETRIC PROOF TEST

ANSWER ALL QUESTIONS TIME: 2 HOURS 50 MARKS

This test consists of two sections. Section A contains questions on geometric proof questions on cone and cylinder whereas Section B consists of questions on the relationship between cone and cylinder

SECTION A

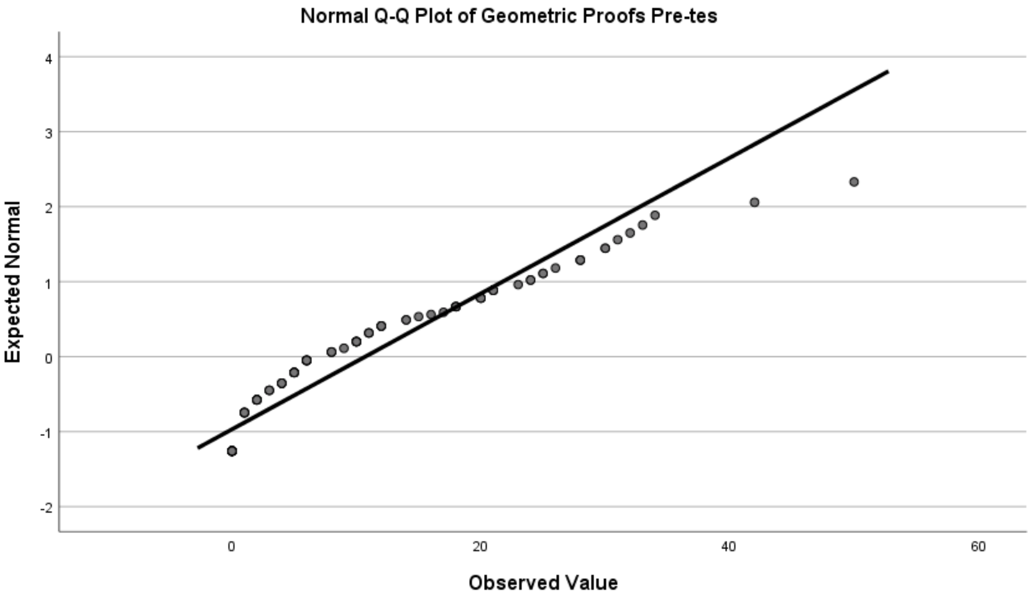
1. Prove that the lateral surface area of a cone is equal to half the product of its slant height and its base circumference
2. prove that the volume of a cone is equal to one-third the product of its base area and its height.
3. Given two cones with the same height, prove that their volumes are proportional to the cubes of their radii?
4. prove that the total surface area of a cone is equal to the sum of its base area and its lateral surface area
5. Proof that the radius of a circular base of a cone (r)= $\frac{\Theta}{360}$R

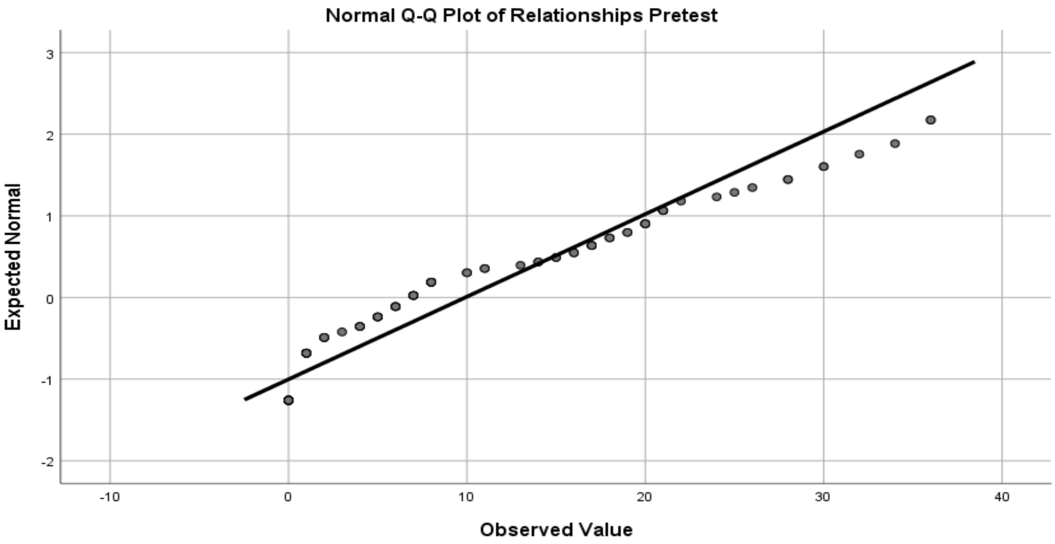
SECTION B

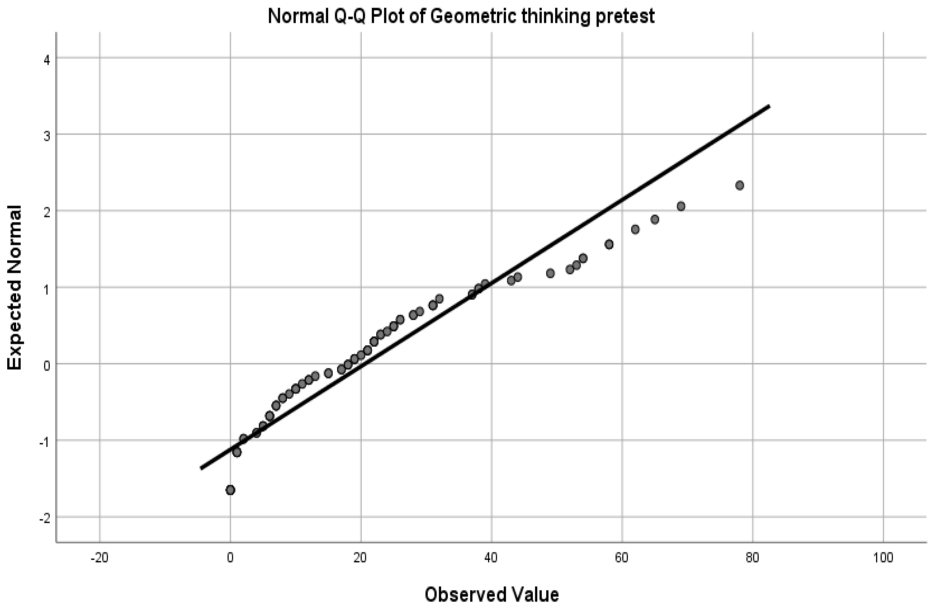
6. Show that the volume of a cone is one third of the volume of a cylinder when they have the same radius and height.
7. Given a cylinder with radius r and height h, what is the ratio of the volume of a cone with the same base and height to the volume of the cylinder?
8. If a cone is inscribed inside a cylinder such that the base of the cone lies on the base of the cylinder, what is the ratio of the volume of the cone to the volume of the cylinder?
9. Given a cylinder with radius r and height h, what is the maximum volume of a cone that can be inscribed inside the cylinder?
10. Given a cone with radius r and height h, what is the maximum volume of a cylinder that can be circumscribed around the cone?

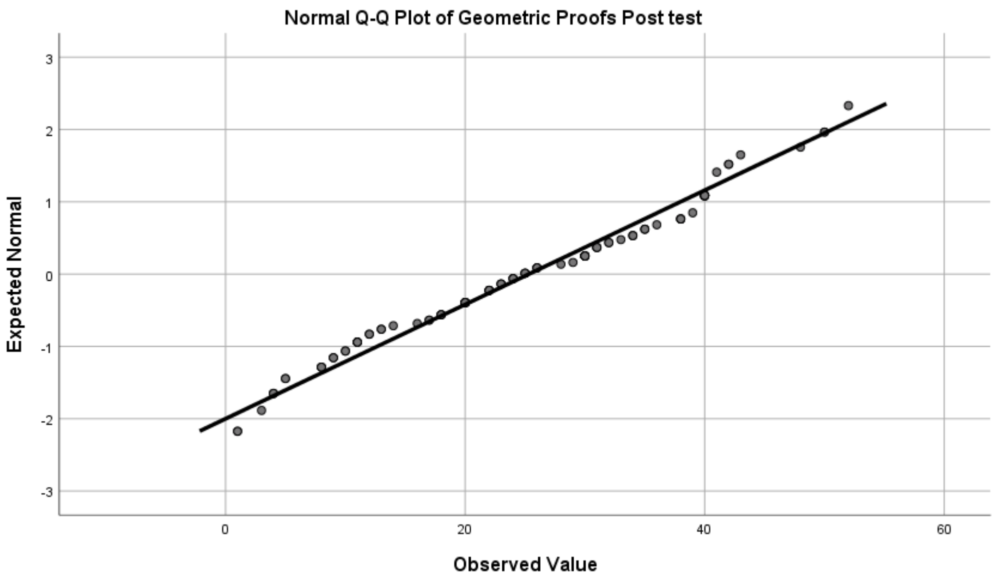
APPENDIX D

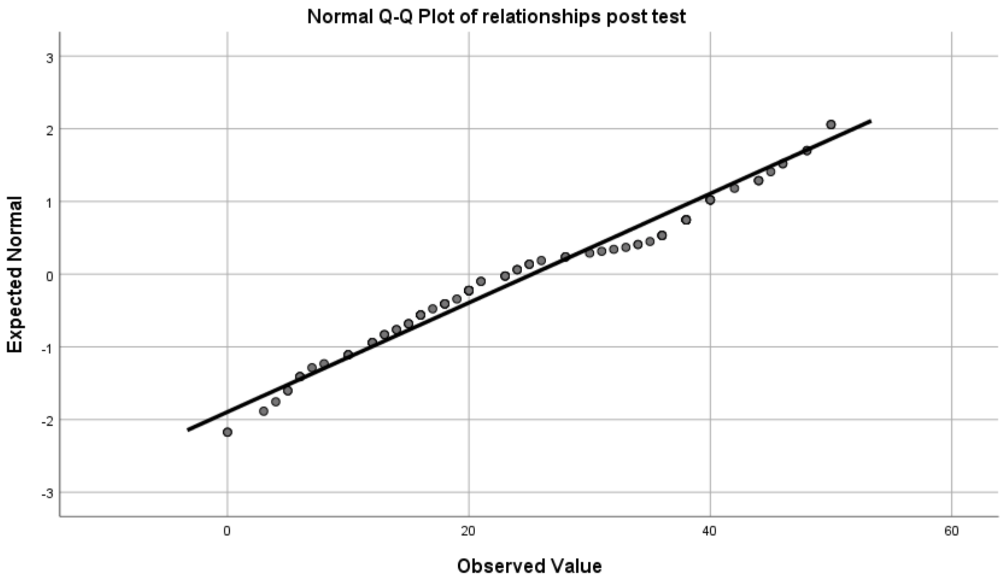
NORMALITY TEST

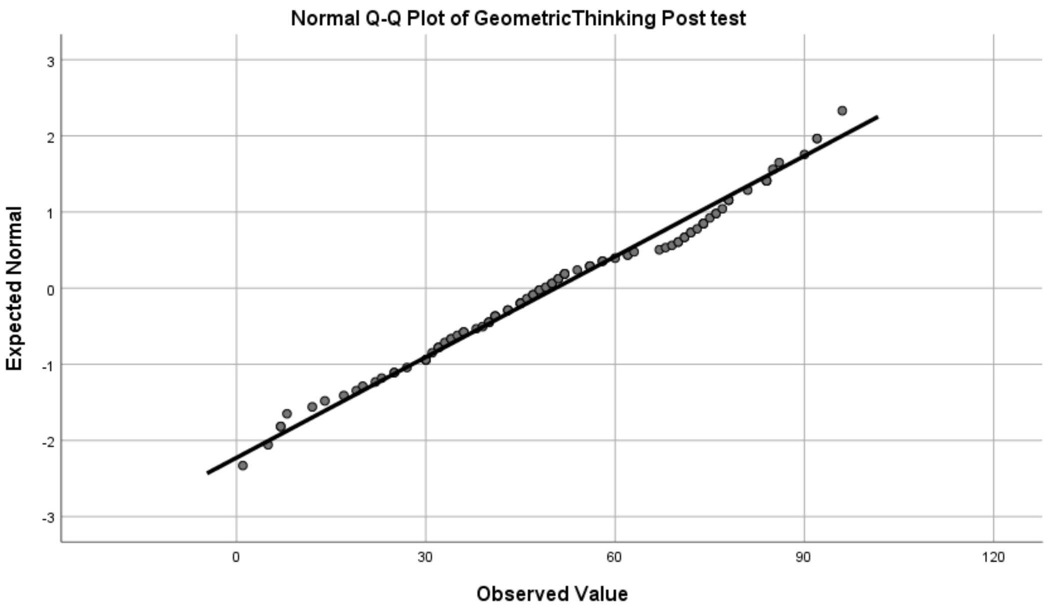





Normal Q-Q Plot of Geometric thinking pretest
Expected Normal
Observed Value
4
3
2
1
0
-1
-2
-20
0
20
40
60
80
100


Normal Q-Q Plot of Geometric Proofs Post test
Expected Normal
Observed Value
3
2
1
0
-1
-2
-3
0
20
40
60


Normal Q-Q Plot of relationships post test
Expected Normal
Observed Value
3
2
1
0
-1
-2
-3
0
20
40
60


Normal Q-Q Plot of GeometricThinking Post test
Expected Normal
Observed Value
3
2
1
0
-1
-2
-3
0
30
60
90
120